INTERFEROMETRIC MAPPING USING THE HARTLEY TRANSFORM

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Sky mapping as practised in radio astronomy produces the most highly resolved images at present available in any field of imaging and also permits very high dynamic range. Consequently the computing load is heavy and any relief from this load would be helpful.

Interferometric mapping is usually described in terms of a measurement procedure using pairs of antennas which yield values of complex coherence corresponding to the vector displacement of the antenna pair. The set of values of complex coherence (or complex visibility) is then regarded as a two-dimensional input function to a Fourier transformation. The result of the transformation is a real two-dimensional function which, after some further treatment according to circumstances, is the output map.

A good part of the development leading to the basic Fourier transform relationship took place in Australia; for the history, see "Imaging Theory in Australia in the Fifties," in Woody Sullivan's book.

The recent activity over the theory of the Hartley transform has confirme this technique as an efficient means of doing spectral analysis on a computer. Consequently, it is timely to ask whether there is an application to interferometric mapping that would enable the efficiency of the Hartley method to be realized.

Two concerns have been raised that suggest that the factor of two speedup gained in the inner loops of the algorithm might not be available in the practice of sky mapping by interferometry.

The first of these concerns is neatly expressed by quoting it in the form in which it first came to my attention. "It's a pity that the Hartley transform does not generalize to two dimensions. Unfortunately, cas(A+B) is not separable into a product as is the case with exp[i(A+B)]; the fact that the two-dimensional Fourier kernel can be written as the product exp(iA)exp(iB) is what permits the two-dimensional transform to be done as a set of 2N calls to the one-dimensional FFT."

This concern has been laid to rest by the discovery of a means of circumventing the nonseparability. See "Fast Two-Dimensional Hartley Transform," by R.N. Bracewell, O. Buneman, H. Hao, and J. Villasenor, to appear in Proc. IEEE, for the method and for other references. A preprint is attached.

The other concern is that the Hartley transform gets its advantage from the property that the input is real, whereas in interferometry the input is complex; therefore there is a feeling among those who have not yet gained familiarity with Hartley methods that they might not be applicable to interferometric mapping.

From a philosophical standpoint one can see that the argument is shaky. After all we never in fact make complex measurements; all measured quantities are real. Complex quantities are a product of the mind rather than of Nature. Since it is very convenient to think with complex numbers, as has been well known to students ever since the introduction of j into alternating current theory by Kelvin, we find it natural to think in terms of the Fourier transform, which is by definition complex.

If one looks at the actual operations performed by the radio astronomy instrumentation it is apparent that the modulus and phase of the complex visibility are certainly not formed; what we do is to multiply by in-phase and quadrature reference oscillations and generate two baseline-dependent numbers that we regard as the real and imaginary parts of a complex quantity, the complex visibility, which we are comfortable with theoretically. But of course the imaginary part of a complex quantity is real and we are not obliged to apply complex number theory to the number pair. It is perfectly open to us to construct, on the baseline plane, the real quantity whose Hartley transform in two dimensions will directly give us the same output map as could have been obtained by the familiar complex method.

The conclusion then is that one really can get the Hartley advantage in computing two-dimensional maps from interferometric data. Most of the refinements of complex Fourier computing are already available for use, including the radix-4 program and FORTRAN assembler code for the VAX (also for the CRAY if you can get at one). In addition there are the new fast permutation and fast rotation algorithms, which are applicable equally to the Fourier and Hartley transforms, but which do not form part of current packages, and therefore will require special attention.

It would be desirable to look carefully at hardware designed for Fourier transformation to make sure that the recently discovered advantages are being obtained.

Fast Two-dimensional Hartley Transforms – R.N. Bracewell, O. Buneman, H. Hao, and J. Villasenor, Space, Telecommunications and Radioscience Laboratory, Stanford University, Stanford, CA94305.

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Abstract – The fast Hartley transform algorithm introduced in 1984 offers an alternative to the fast Fourier transform, with the advantages of not requiring complex arithmetic or a sign change of i to distinguish inverse transformation from direct. A two-dimensional extension is described that speeds up Fourier transformation of real digital images.

The two-dimensional discrete Hartley transform of the function $f(x,y)$ is defined [1], [2] by analogy with Hartley's integral transform [3], by

$$H(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \operatorname{cas}[2\pi(ux/M + vy/N)],$$

where $\operatorname{cas}\theta = \cos\theta + \sin\theta$. Since the one-dimensional Hartley transform offers a speed advantage [4], [5], [6], [7] over the Fast Fourier Transform (FFT) for numerical spectral analysis, it might be expected that the two-dimensional Hartley transform would offer similar advantages in image processing and other applications of multidimensional spectral analysis.

Two-dimensional fast Fourier transformation is performed by calling a succession of one-dimensional FFTs: first one transforms all the rows using the kernel $\exp(i2\pi ux/M)$ and then transforms column by column using $\exp(i2\pi vy/N)$. The result amounts to transforming with the product kernel $\exp[i2\pi(ux/M + vy/N)]$, which is the two-dimensional Fourier kernel, and may be thought of as representing a wave at some general angle to the two coordinate axes.

However, the kernel $\operatorname{cas}[2\pi(ux/M + vy/N)]$, unlike $\exp[i2\pi(ux/M + vy/N)]$, is not separable into a product of factors. This letter is to report success in overcoming this apparent difficulty with generalizing the Hartley transform to more than one dimension. If one were to follow the row and column procedure using one-dimensional Hartley transforms, the effective two-dimensional transform kernel would be the product $\operatorname{cas}(2\pi ux/M) \times \operatorname{cas}(2\pi vy/N)$, which has no simple physical interpretation as an oblique wave. Instead, one wants the transform kernel $\operatorname{cas}[2\pi(ux/M + vy/N)]$, an expression which, as in the Fourier case, represents a wave, with u and v being the components of a vector which is perpendicular to the wave fronts and whose magnitude is the inverse of the wavelength.

One way of doing this is to try to apply the same steps in two dimensions that led to the fast transform in one dimension. The approach involves progressively quartering the data until 2×2 cells are reached, and applying the two-dimensional Hartley shift theorem [1] to synthesize the full-size transform from the transforms of the elementary cells. It is necessary to study the phenomenon of two-dimensional permutation, which generalizes from bit reversal in one dimension, in order that the results appear in the correct spatial order. Either prepermutation or postpermutation may be adopted. A program has been developed using prepermutation, which corresponds to "decimation in time," a term that is inappropriate in two dimensions.

A second way is to take Hartley transforms of the rows to form an intermediate array whose rows are split into even and odd parts. Two further arrays are then formed; one consists of the Hartley transforms of the columns of the even array, the other of the Hartley transforms of the columns of the odd array, written in reverse order bottom to top, except for the top row. These two arrays are then added. The program was developed using postpermutation, along the lines of the "decimation in frequency" form [8] of the one-dimensional Hartley algorithm .

A third method begins by direct analogy with the two-dimensional FFT, takes the one-dimensional discrete Hartley transforms of the rows one by one and then transforms the columns. The temporary outcome $T(u,v)$ is of the form

$$T(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \operatorname{cas}(2\pi ux/M) \operatorname{cas}(2\pi vy/N)$$

which, as mentioned above, is not the Hartley transform. However, the result can be converted to the desired two-dimensional Hartey transform by a trivial step. First we note the trigonometrical identity

$$2\operatorname{cas}(\alpha+\beta) = \operatorname{cas}\alpha \operatorname{cas}\beta + \operatorname{cas}\alpha \operatorname{cas}(-\beta) + \operatorname{cas}(-\alpha)\operatorname{cas}\beta - \operatorname{cas}(-\alpha)\operatorname{cas}(-\beta).$$

Let the data array have dimensions $M \times N$. Then the desired Hartley transform $H(u,v)$ can be expressed as a sum of four temporary transforms

$$2H(u,v) = T(u,v) + T(M-u,v) + T(u,N-v) - T(M-u,N-v)$$

$$= A + B + C - D.$$

The desired transform can be obtained from $T(u,v)$ by combining four members of $T(u,v)$ situated on the vertices of a rectangle. To compute the combination efficiently we work on sets of four values at a time, in each set forming the diagonal excess $E = \frac{1}{2}[(A+D) - (B+C)]$. Then four replacement statements

$$A \leftarrow A - E, \quad B \leftarrow B + E, \quad C \leftarrow C + E, \quad D \leftarrow D - E$$

convert $T(u, v)$ to $H(u, v)$. The quantity E is zero where $u = 0$, $v = 0$, $u = \frac{1}{2}M$, $v = \frac{1}{2}N$; consequently the replacement need only be made for values of u from 1 to $\frac{1}{2}M - 1$ and for v from 1 to $\frac{1}{2}N - 1$. This discussion shows that the two-dimensional discrete Hartley transform is speedily calculable by $M + N$ calls to a one-dimensional Hartley routine followed by some additions. With an 8×8 array the extra additions add about 8 per cent to the operations count and, as array size increases, the fraction of extra operations diminishes.

Once the Hartley transform is arrived at, the real and imaginary parts of the Fourier transform could be obtained by rotating the Hartley array half a turn and adding and subtracting respectively. But it is apparent that, where the Fourier transform is the sole objective, it is not necessary to go via the the full two-dimensional Hartley transform; it suffices to calculate the temporary transform $T(u, v)$. With the definitions introduced above, and with $u \leq \frac{1}{2}M, v \leq \frac{1}{2}N$, one then gets the Fourier transform at (u, v) as $\frac{1}{2}(B+C) - i\frac{1}{2}(A-D)$ and, correspondingly, that at $(M-u, v)$ as $\frac{1}{2}(A+D) - i\frac{1}{2}(B-C)$. The Fourier transforms at $(u, N-v)$ and at $(M-u, N-v)$ follow from the hermiticity.

Various doctored codes derived from the fast Fourier transform are available [7] that may be as fast as the fast Hartley FHT, but are incapable of inverting their own output; two different one-legged programs of this type are needed to equal one fast Hartley program.

It is thus now possible to compute two-dimensional Fourier transforms without the use of complex arithmetic and by means of one algorithm that is the same whether one is transforming or retransforming.

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FOURIER ANALYSIS OF IMAGE BY ONE-DIMENSIONAL HARTLEY TRANSFORMS

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