Better redundant configurations for the AT.

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Because of other pressures the original redundant configurations for the AT were chosen without a full investigation of error propagation. It now turns out that after a few days observations with these configurations, error propagation becomes a serious problem. This is illustrated by Table 1. On the left side of this table are listed the aerial positions for each of 10 days, for the original 3km redundant array. Below each aerial position is the variance of the estimate of the phase error for that aerial. On the right side of the table are listed the lengths of the spacings observed, and below each the variance of the estimate of the spacing phase. A * beside a spacing indicates that the same spacing was measured twice on this day. A ^ beside a spacing indicates that that spacing was also measured on a previous day.

The variances are estimated by assuming that all measured phases have equal, unit variance. The mathematical details of this calculation are given in the Appendix. A variance of 9 means an error 3 times the measurement error, and a variance of 43 means that the error estimate is nearly 7 times the measurement error. Table 1 shows that after 5 days the errors are quite large and the situation continues to deteriorate as more days are added.

The solution is to allow a little more redundancy. Table 2 illustrates what can be achieved with the existing station locations, for a 3-km 6-day sequence. After a least-squares fit to the observational data (see Appendix), the variance of all deduced quantities is now less than 4, i.e. the errors are less than double the measurement errors. This significant improvement has been achieved at a very modest cost: the new 6-day scheme measures 33 different spacings, whereas the original scheme measured 37 different spacings in 6 days. ( A non-redundant sequence measures 60 uncalibrated spacings in the same time ).

This particular solution is not optimum in any sense; it has been chosen to give acceptable error levels and reasonably uniform coverage. It seems probable that longer sequences can be found which retain these two essential properties, and that similar 1.5-km and 6-km sequences can be found, using the existing station locations. However significant computing effort will be required to do this. In the mean time the solution already found can be used to
compare the performance of the redundant scheme with a non-redundant scheme, calibrated by using self-cal.

These results do not modify the requirements for the station locations. The station locations were selected to include the positions needed for the first three days of a redundant sequence, but the requirement of a redundant observing sequence did not otherwise affect the selection of stations. The first two days of the original redundant sequence are retained in the present solution, and indeed are essential to any redundant configuration. Therefore the reasons for selecting the present set of stations are still valid.
Appendix

Evaluation of phases using over-determined, redundant arrays.

Notation:

From observations on day \( d \) with a pair of aerials \( i \) and \( j \) at east-west positions \( x_{d,i} \) and \( x_{d,j} \), we determine the measured phase \( \phi_{d,ij} \), which is related to the aerial phase errors \( e_{d,i} \), \( e_{d,j} \), and the true phase due to the sky \( \psi(s_{d,ij}) \) where the spacing \( s_{d,ij} = x_{d,i} - x_{d,j} \)

\[
\phi_{d,ij} = \psi(s_{d,ij}) + e_{d,i} - e_{d,j} \quad . \quad . \quad (1)
\]

Constraints

From a number of such observations we obtain a set of linear equations relating the observations, \( \{\phi_{d,ij}\} \) to the quantities that we need to know, \( \{\psi(s_{d,ij})\} \) and the errors, \( \{e_{d,i}\} \).

This set of equations will be singular unless we add constraints which serve to define the zero phase for each day, and fix the slope of the errors for a series of days. For example, for each \( d \), for the zero of phase:

\[
\sum e_{d,i} = 0 \quad . \quad . \quad (2)
\]

and for all \( d \)'s taken to-gether, for the slope:

\[
\sum \sum x_{d,i} e_{d,i} = 0 \quad . \quad . \quad (3)
\]

Solution - minimal redundancy

Writing equations (1), (2), and (3) to-gether in matrix notation, we have

\[
A \ x = b \quad . \quad . \quad (4)
\]

where the vector \( x \) is defined as:

\[
x^T = (\psi(s_{1,12}), \psi(s_{1,13}), \ldots, e_{1,1}, e_{1,2}, \ldots)
\]

and \( b \) by:
\( b' = (\phi_{1,2}, \phi_{1,3}, \ldots) \)

and \( A \) is a matrix which contains only +1, -1, or 0, except in the row which represents equation (3).

For the original redundant configuration the redundancy was chosen carefully to ensure that \( A \) was square and non-singular, i.e. \( A^{-1} \) exists, and

\[
x = A^{-1} b \quad \ldots \quad (5)
\]

**Solution - extra redundancy**

When extra redundancy is introduced, equations (4) are overdetermined, provided the configurations are chosen sensibly. We then seek a least squares solution for \( x \):

\[
(A' x - b')' (A' x - b) = \text{minimum}
\]

\[
A' A x = A' b
\]

\[
x = (A' A)^{-1} A' b \quad \ldots \quad (6)
\]

Here \((A' A)\) is a square matrix, and should be non-singular.

\(A'^{-1} = (A' A)^{-1} A'\) is known as the generalised inverse. There is a NAG subroutine, \texttt{F01BLF}, which calculates this generalised inverse, and detects any problems, such as singularity.

**Variance of solution values.**

The variance of an element \( x_i \) of \( x \) can be calculated from (6):

\[
\sigma_i^2 = \sum (A_{i,j} A'^{-1})^2 v_j
\]

where \( \sigma_i^2 \) is the variance of \( x_i \), and \( v_j \) is the variance of \( b_j \).
TABLE 2

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Note: The table continues with similar entries and is part of a larger document.