CSIRO DIVISION OF RADIOPHYSICS

28 June 1983

AT FILE NOTE
~~AT/05.2/035~~
RNM

POSSIBLE STATION LOCATIONS FOR COMPACT ARRAY

The table below gives station positions computed by Geoff Poulton for the 6 km compact array in a configuration based on 20 metre increments for 1.5, 3 and 6 km baselines. This layout is interim in a sense as we are looking at further optimization of the configuration perhaps involving non-grating arrays. It does, however, represent a practical solution in that future solutions are likely to have a similar number of and overall distribution of stations - in particular the main concentration at the western end of the 3 km track and two stations at the 6 km point.

Stn.	Intervals (20m)	Distance (m)	Stn.	Intervals (20m)	Distance (m)	Stn.	Intervals (20m)	Distance (m)
1	0	0	13	77	1540	25	124	2480
2	2	40	14	79	1580	26	125	2500
3	4	80	15	81	1620	27	134	2680
4	8	160	16	83	1660	28	135	2700
5	20	400	17	88	1760	29	138	2760
6	24	480	18	94	1880	30	140	2800
7	30	600	19	96	1920	31	142	2840
8	41	820	20	97	1940	32	146	2920
9	50	1000	21	102	2040	33	148	2960
10	57	1140	22	106	2120	34	149	2980
11	64	1280	23	112	2240	35	150	3000
12	76	1520	24	118	2360	36	296	5920
						37	300	6000

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ON DIFFERENCE TRIANGLES AND INVERSE-ZOOM ARRAYS

P.R. Wild - 1 December 1983

1. INTRODUCTION

The $\binom{n}{2}$ differences between antennas in a linear configuration of n antennas may be represented in a triangular form, known as the *difference triangle*, which has certain arithmetic properties. We use these properties to consider the possibility of inverse-zoom arrays - a sequence of configurations which yields differences which become progressively closer together as we move from the small differences to the larger ones.

We conclude that for $n \geqslant 6$, an inverse-zoom array is not possible. For $n = 4,5$ an inverse-zoom array (if it exists) must be close to a regular full-fill grating array. For $n = 3$ an inverse-zoom array may well exist. For $n = 2$ any set of differences is possible.

2. DIFFERENCE TRIANGLES AND THEIR ARITHMETIC PROPERTIES

Consider a linear array with n antennas. Let the station location of the antennas on day i be given by integers $a_{i1}, \ldots, a_{in}$ (i=1, ..., m). Write $d_{ik}^{j} = a_{ik+j} - a_{ik}$ for the distance between the $(k+j)^{th}$ and k^{th} antennas $1 \leqslant j \leqslant n-1$; $1 \leqslant k \leqslant n-j$. We may represent the distances in triangular form (the difference triangle):

$$
\begin{array}{ccccccccccc}
 & & & & & d_{i1}^{n-1} & & & & & \\
 & & & & d_{i1}^{n-2} & & d_{i2}^{n-2} & & & & \\
 & & & \cdot & & & & \cdot & & & \\
 & & \cdot & & & & & & \cdot & & \\
 & & d_{i1}^{3} & \cdot & \cdot & & \cdot & \cdot & d_{in-3}^{1} & & \\
 & d_{i1}^{2} & & d_{i2}^{2} & \cdot & \cdot & \cdot & \cdot & & d_{in-2}^{1} & \\
d_{i1}^{1} & & d_{i2}^{1} & & d_{i3}^{1} & \cdot & \cdot & \cdot & \cdot & & d_{in-1}^{1}
\end{array}
$$

$$
\begin{aligned}
d_{ik}^{j} = a_{i\,k+j} - a_{ik} &= (a_{i\,k+j} - a_{i\,k+j-1}) + (a_{i\,k+j-1} - a_{i\,k+j-2}) + \ldots \\
&\quad \ldots (a_{i\,k+2} - a_{i\,k+1}) + (a_{i\,k+1} - a_{ik}) \\
&= d_{ik+j-1}^{1} + d_{ik+j-2}^{1} + \ldots + d_{ik+1}^{1} + d_{ik}^{1} \\
&= \sum_{\ell=k}^{k+j-1} d_{i\ell}^{1}
\end{aligned}
$$

i.e., the k^{th} element of row j = the sum of j consecutive elements on the bottom row of the difference triangle starting at d^1_{ik}.

For example:

$$\begin{matrix} & a_{11} & a_{12} & a_{13} & a_{14} \\ = & 0 & 1 & 4 & 6 \end{matrix}$$

has difference triangle

$$\begin{matrix} & & 6 & & \\ & 4 & & 5 & \\ 1 & & 3 & & 2 \end{matrix}$$

row 1 contains the differences between successive antennas

row 2 contains the difference between two antennas separated by one other antenna

- this difference is the sum of the differences between each of the two antennas and the third (separating) one.

The property of a difference triangle which is of interest to us is that:

the sum of the elements in the top $(\frac{n-1}{2})$ rows of a difference triangle

= the sum of the elements in the bottom $(\frac{n-1}{2})$ rows of that difference triangle.

Let $j \leqslant \frac{n-1}{2}$.

$$\sum_{\ell=1}^{n-j} d^j_{i\ell} = \text{sum of } j^{th} \text{ row of difference triangle}$$

$$= \sum_{\ell=1}^{n-j} \sum_{k=\ell}^{\ell+j-1} d^1_{ik}$$

$$= \sum_{k=1}^{n-1} \alpha_k \, d^1_{ik}$$

where $\alpha_k = k$ if $1 \leqslant k \leqslant j$

$= j$ if $j \leqslant k \leqslant n-j$

$= n-k$ if $n-j \leqslant k \leqslant n-1$

$$\sum_{\ell=1}^{j} d^{n-j}_{i\ell} = \text{sum of } (n-j)^{th} \text{ row of difference triangle}$$

$$= \sum_{\ell=1}^{j} \sum_{k=\ell}^{\ell+n-j-1} d^1_{ik}$$

$$= \sum_{k=1}^{n-1} \beta_k \; d_{ik}^{1}$$

where $\beta_k = k$ if $1 \leqslant k \leqslant j$
$= j$ if $j \leqslant k \leqslant n-j$
$= n-k$ if $n-j \leqslant k \leqslant n-1$

$$\text{Thus} \quad \sum_{\ell=1}^{n-j} d_{i\ell}^{j} = \sum_{\ell=1}^{j} d_{i\ell}^{n-j}$$

i.e., sum of j^{th} row = sum of $(n-j)^{th}$ row for $j=1, \ldots [\frac{n-1}{2}]$.

$$\text{Hence} \quad \sum_{k=1}^{[\frac{n-1}{2}]} \sum_{\ell=1}^{n-k} d_{i\ell}^{k} = \sum_{k=1}^{[\frac{n-1}{2}]} \sum_{\ell=1}^{k} d_{i\ell}^{n-k}$$

i.e., sum of bottom $[\frac{n-1}{2}]$ rows = sum of top $[\frac{n-1}{2}]$ rows.

3. INVERSE-ZOOM ARRAYS

Put $p = [\frac{n-1}{2}]$.

There are $\frac{1}{2}p(p+1)$ elements in the top p rows and

$\frac{1}{2}p(2n-1-p)$ elements in the bottom p rows.

Since $n \sim 2p$ there are approximately three times as many elements in the bottom half of a difference triangle as in the top half. Since, in general, the bottom half contains the smallest and the top half the largest differences (their sums being equal) it is clear that for every large difference we have several smaller differences. It follows that an 'inverse-zoom' array, in which the large differences outnumber the small ones, cannot be realized except perhaps by having redundancies among the small differences. The calculations which follow make precise the above claim when $n \geqslant 6$. For $n = 4$ and 5 the calculations suggest that perhaps an inverse-zoom array may exist provided it is close to a regular full-fill array (the border-line between zoom arrays and inverse zoom arrays).

Consider the following possibility:

We have m configurations $a_{i1}, \ldots, a_{in}$ $i=1, \ldots, m$; of n antennas, giving distinct differences (or at least very few redundancies) which make up an inverse-zoom array. That is, the differences get progressively closer together as we go from the small differences to the larger ones.

Put $p = [\frac{n-1}{2}]$. Let the average distance between successive differences in the first $S = \frac{m}{2}[\frac{n-1}{2}](2n-1-[\frac{n-1}{2}]) = \frac{mp}{2}(2n-1-p)$ differences be f_1. Let the largest distance between successive differences in the last $T=\frac{m}{2}\ p(p+1)$ differences be f_2. We have $f_1 > f_2$ for an inverse-zoom array.

Now $\sum_{i=1}^{S} f_1 i \leqslant$ sum of first S differences $\leqslant \sum_{i=1}^{m} \sum_{k=1}^{p} \sum_{\ell=1}^{n-k} d_{i\ell}^{k}$

= sum of terms in bottom p rows of the m difference triangles.

Also, sum of terms in top p rows of the m difference triangles

$$= \sum_{i=1}^{m} \sum_{k=1}^{p} \sum_{\ell=1}^{k} d_{i\ell}^{n-k} \leqslant \text{sum of last T differences}$$

$$\leqslant \sum_{i=1}^{T} Sf_1 + R + f_2 i$$

where R = difference between the $(\frac{n}{2})$-T difference and the S difference.

Thus $\sum_{i=1}^{S} f_1 i \leqslant \sum_{i=1}^{T} Sf_1 + R + f_2 i$

Now $\sum_{i=1}^{S} f_1 i = \frac{1}{2} F_1 S(S+1)$

and $\sum_{i=1}^{T} Sf_1 + R + f_2 i = f_1 ST + RT + \frac{1}{2} f_2 T(T+1)$.

If n is odd then n = 2p+1 and we have

$$S = \frac{m}{2} p(3p+1), \quad T = \frac{m}{2} p(p+1), \quad R = 0 \ .$$

So, $\frac{1}{2} f_1 \frac{m}{2} p(3p+1) (\frac{m}{2} p(3p+1) + 1)$

$$\leqslant f_1 \frac{m}{2} p(3p+1) \frac{m}{2} p(p+1) + \frac{1}{2} f_2 \frac{m}{2} p(p+1) (\frac{m}{2} p(p+1) + 1)$$

$$\therefore f_1 [\frac{m^2}{8} (9p^4 + 6p^3 + p^2) + \frac{m}{4} (3p^2 + p)]$$

$$\leqslant f_1 [\frac{m^2}{4} (3p^4+4p^3+p^2)] + f_2 [\frac{m^2}{8}(p^4+2p^3+p^2) + \frac{m}{4}(p^2+p)]$$

$$\therefore \; f_1 \left[\frac{m^2}{8}(3p^4-2p^3-p^2) + \frac{m}{4}(3p^2+p)\right]$$

$$\leqslant \; f_2 \left[\frac{m^2}{8}(p^4+2p^3+p^2) + \frac{m}{4}(p^3+p)\right] \; .$$

It follows that for $p \geqslant 3$ we have $f_2 > f_1$ contradicting our assumption that we have an inverse-zoom array. For p=2 and large m we have $f_2 \geqslant \frac{7}{9} f_1$, which means that for n=5 perhaps an inverse-zoom array is possible provided it is not too 'zoomy'.

For p=1 (n=3) we require $f_2 \geqslant \frac{f_1}{m}$, so an inverse-zoom array seems possible for large m.

If n is even then n = 2p+2 and we have

$$S = \frac{3m}{2} \; p(p+1), \quad T = \frac{m}{2} p(p+1), \quad R \approx m(p+1) f_2 \quad .$$

So $\frac{1}{2} f_1 \frac{3m}{2} p(p+1) \left[\frac{3m}{2} p(p+1) + 1\right]$

$$\leqslant \; f_1 \; \frac{3m^2}{4} \; p^2 \; (p+1)^2 + \frac{m^2}{2} p(p+1)^2 \; f_2 + \frac{1}{2} f_2 \frac{m}{2} p(p+1) \left[\frac{m}{2} p(p+1) + 1\right] \; .$$

$$\therefore \; f_1 \left[\frac{3m^2}{8} (p^4+2p^3+p^2) + \frac{3m}{4} p(P+1)\right]$$

$$\leqslant \; f_2 \left[\frac{m^2}{8} (p^4+6p^3+9p^2+4p) + \frac{m}{4} p \; (p+1)\right] \; .$$

This contradicts $f_2 < f_1$ if $p \geqslant 2$.

For p=1 (n=4) we have (for large m) $f_2 \geqslant \frac{3}{5} f_1$.

So for n=4 an inverse-zoom array may be possible for a small zoom factor.

For n=2 anything is possible.