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Miscellaneous notes on the derivation of some formulæ and special conditions in FITS WCS Paper II

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Abstract. Background notes on the mathematical derivation of some troublesome formulæ and special conditions in FITS WCS Paper II.

1. Spherical coordinate transformation

Derivation of special conditions for the spherical coordinate transformation between native and celestial coordinates.

In the following, bear in mind that $(\phi_{\rm p}, \theta_{\rm p})$ and $(\alpha_{\rm p}, \delta_{\rm p})$ refer to *different* points; the common "p" subscript indicates that they are coordinates of "the pole", but not the same pole; $(\phi_{\rm p}, \theta_{\rm p})$ are the native coordinates of the *celestial pole*, and $(\alpha_{\rm p}, \delta_{\rm p})$ are the celestial coordinates of the *native pole*. Generally the native and celestial poles do not coincide.

On the other hand, (ϕ_0, θ_0) and (α_0, δ_0) do refer to the same, fiducial, point (normally the reference point of the projection).

The spherical coordinate transformation equations from Paper II are:

$$\begin{aligned} \alpha &= \alpha_{\rm p} + \arg(\sin\theta\cos\delta_{\rm p} - \cos\theta\sin\delta_{\rm p}\cos(\phi-\phi_{\rm p}), \\ &\quad -\cos\theta\sin(\phi-\phi_{\rm p})), \\ \delta &= \sin^{-1}(\sin\theta\sin\delta_{\rm p} + \cos\theta\cos\delta_{\rm p}\cos(\phi-\phi_{\rm p})), \end{aligned} \quad (1)$$

and their inverses

$$\begin{aligned} \phi &= \phi_{\rm p} + \arg(\sin\delta\cos\delta_{\rm p} - \cos\delta\sin\delta_{\rm p}\cos(\alpha-\alpha_{\rm p}), \\ &\quad -\cos\delta\sin(\alpha-\alpha_{\rm p})), \\ \theta &= \sin^{-1}(\sin\delta\sin\delta_{\rm p} + \cos\delta\cos\delta_{\rm p}\cos(\alpha-\alpha_{\rm p})). \end{aligned} \quad (2)$$

Relations between (ϕ_0, θ_0), (α_0, δ_0), $(\phi_{\rm p}, \theta_{\rm p})$, and $(\alpha_{\rm p}, \delta_{\rm p})$ are derived from these. ϕ_0, θ_0, α_0, δ_0, and $\phi_{\rm p}$, are considered to be *given* while $\theta_{\rm p}$, $\alpha_{\rm p}$, and $\delta_{\rm p}$ are considered to be *derivative*.

1. $\theta_{\rm p} = \delta_{\rm p}$.

   Proof: substituting $\delta = +90°$ into Eq. (2θ) gives $\sin\theta_{\rm p} = \sin\delta_{\rm p}$, whence $\theta_{\rm p} = \delta_{\rm p}$.

   Comment: the native latitude of the celestial pole is always equal to the celestial latitude of the native pole. This is a basic property of spherical coordinate rotations.

2. If $\delta_0 = +90°$:

   - $\delta_{\rm p} = \theta_0$.

     Proof: substituting $(\alpha, \delta) = (\alpha_0, 90°)$ into Eq. (2θ) reduces it to $\sin\theta_0 = \sin\delta_{\rm p}$, whence $\delta_{\rm p} = \theta_0$.

   - $\phi_{\rm p} = \phi_0 \quad \ldots \theta_0 \neq \pm 90°$.

     Proof: substituting $(\alpha, \delta) = (\alpha_0, 90°)$ and $\delta_{\rm p} = \theta_0$ into Eq. (2ϕ) gives $\phi_0 = \phi_{\rm p} + \arg(\cos\theta_0, 0)$. For $\theta_0 \neq \pm 90°$ this reduces to $\phi_{\rm p} = \phi_0$.

   - $\alpha_{\rm p}$ is indeterminate.

     Proof: substituting $(\phi, \theta) = (\phi_0, \theta_0)$, $\delta_{\rm p} = \theta_0$, and $\phi_{\rm p} = \phi_0$ $(\theta_0 \neq \pm 90°)$ into Eq. (1α) and rearranging gives $\alpha_{\rm p} = \alpha_0 - \arg(0, 0)$.

     Now, if $\theta_0 = \pm 90°$, substituting $(\phi, \theta) = (\phi_0, \pm 90°)$ and $\delta_{\rm p} = \theta_0 = \pm 90°$ into Eq. (1α) gives $\alpha_0 = \alpha_{\rm p} + \arg(0, 0)$.

   Comment: for $\delta_0 = 90°$ the celestial pole is at the fiducial point. Therefore, unless the fiducial point is at one of the native poles, the native longitude of the celestial pole, $\phi_{\rm p}$, must be equal to the native longitude of the fiducial point, ϕ_0; if it is given (via `LONPOLE`a) as some other value then the FITS WCS header is invalid.

3. If $\delta_0 = -90°$:

   - $\delta_{\rm p} = -\theta_0$.

     Proof: substituting $(\alpha, \delta) = (\alpha_0, -90°)$ into Eq. (2θ) reduces it to $\sin\theta_0 = -\sin\delta_{\rm p}$, whence $\delta_{\rm p} = -\theta_0$.

   - $\phi_{\rm p} = \phi_0 + 180° \quad \ldots \theta_0 \neq \pm 90°$.

     Proof: substituting $(\alpha, \delta) = (\alpha_0, 90°)$ and $\delta_{\rm p} = -\theta_0$ into Eq. (2ϕ) gives $\phi_0 = \phi_{\rm p} + \arg(-\cos\theta_0, 0)$. For $\theta_0 \neq \pm 90°$ this reduces to $\phi_0 = \phi_{\rm p} + 180°$.

   - $\alpha_{\rm p}$ is indeterminate.

     Proof: substituting $(\phi, \theta) = (\phi_0, \theta_0)$, $\delta_{\rm p} = -\theta_0$, and $\phi_{\rm p} = \phi_0 + 180°$ $(\theta_0 \neq \pm 90°)$ into Eq. (1α) and rearranging gives $\alpha_{\rm p} = \alpha_0 - \arg(0, 0)$.

     Now if $\theta_0 = \pm 90°$, substituting $(\phi, \theta) = (\phi_0, \pm 90°)$ and $\delta_{\rm p} = -\theta_0 = \mp 90°$ into Eq. (1α) gives $\alpha_0 = \alpha_{\rm p} + \arg(0, 0)$.

   Comment: for $\delta_0 = -90°$ the celestial pole is antipodal to the fiducial point. Therefore, unless the fiducial point is at

Table 1. Summary of the determination of (α_p, δ_p) for special-case values of θ_0 and δ_0. The three places where ϕ_p appears in the table indicate restrictions on its value for the particular values of θ_0 and δ_0.

	$\theta_0 = +90°$		$\theta_0 = -90°$		$\theta_0 \neq \pm 90°$	
$\delta_0 = +90°$	$\delta_p = \theta_0 = +90°$	...2	$\delta_p = \theta_0 = -90°$	...2	$\delta_p = \theta_0$	...2
	$\delta_p = \delta_0 = +90°$	...4	$\delta_p = -\delta_0 = -90°$	...5	$\phi_p = \phi_0$	...2
	α_p indeterminate	...2, 4, 6	α_p indeterminate	...2, 5, 7	α_p indeterminate	...2
	$\alpha_p \equiv \alpha_0$		$\alpha_p \equiv \alpha_0$		$\alpha_p \equiv \alpha_0$	
$\delta_0 = -90°$	$\delta_p = -\theta_0 = -90°$	...3	$\delta_p = -\theta_0 = +90°$	...3	$\delta_p = -\theta_0$	...3
	$\delta_p = \delta_0 = -90°$	...4	$\delta_p = -\delta_0 = +90°$	...5	$\phi_p = \phi_0 + 180°$	...3
	α_p indeterminate	...3, 4, 7	α_p indeterminate	...3, 5, 6	α_p indeterminate	...3
	$\alpha_p \equiv \alpha_0$		$\alpha_p \equiv \alpha_0$		$\alpha_p \equiv \alpha_0$	
$\delta_0 \neq \pm 90°$	$\delta_p = \delta_0$	...4	$\delta_p = -\delta_0$	...5	If $\delta_p = +90°$, then $\alpha_p = \alpha_0 + (\phi_p - \phi_0) - 180°$	
	$\alpha_p = \alpha_0$	...4	$\alpha_p = \alpha_0 + 180°$	...5	If $\delta_p = -90°$, then $\alpha_p = \alpha_0 - (\phi_p - \phi_0)$	

	$\theta_0 = 0°$	
$\delta_0 = 0°$	$\phi_p = \phi_0 \pm 90°$	...8
	δ_p indeterminate	...8
	$\alpha_p = \alpha_0 - (\phi_p - \phi_0)$	...8
	$= \alpha_0 \mp 90°$	...8

one of the native poles, the native longitude of the celestial pole, ϕ_p, must be antipodal to the native longitude of the fiducial point, ϕ_0; if it is given (via LONPOLEa) as some other value then the FITS WCS header is invalid.

4. If $\theta_0 = +90°$:
   - $\delta_p = \delta_0$.

     Proof: substituting $(\phi, \theta) = (\phi_0, 90°)$ into Eq. (1δ) reduces it to $\sin\delta_0 = \sin\delta_p$, whence $\delta_p = \delta_0$.
   - $\alpha_p = \alpha_0 \quad \ldots\delta_0 \neq \pm 90°$.

     Proof: substituting $(\phi, \theta) = (\phi_0, 90°)$ and $\delta_p = \delta_0$ into Eq. (1α) gives $\alpha_0 = \alpha_p + \arg(\cos\delta_0, 0)$. For $\delta_0 \neq \pm 90°$ this reduces to $\alpha_p = \alpha_0$.
   - α_p is indeterminate if $\delta_0 = \pm 90°$.

     Proof: from above if $\delta_0 = \pm 90°$, $\alpha_0 = \alpha_p + \arg(0, 0)$.

   Comment: for $\theta_0 = 90°$ the fiducial point is at the native pole, so these results are essentially just the definition of (α_p, δ_p) as the celestial coordinates of the native pole.
5. If $\theta_0 = -90°$:
   - $\delta_p = -\delta_0$.

     Proof: substituting $(\phi, \theta) = (\phi_0, -90°)$ into Eq. (1δ) reduces it to $\sin\delta_0 = -\sin\delta_p$, whence $\delta_p = -\delta_0$.
   - $\alpha_p = \alpha_0 + 180° \quad \ldots\delta_0 \neq \pm 90°$.

     Proof: substituting $(\phi, \theta) = (\phi_0, 90°)$ and $\delta_p = -\delta_0$ into Eq. (1α) gives $\alpha_0 = \alpha_p + \arg(-\cos\delta_0, 0)$. For $\delta_0 \neq \pm 90°$ this reduces to $\alpha_0 = \alpha_p + 180°$.
   - α_p is indeterminate if $\delta_0 = \pm 90°$.

     Proof: from above if $\delta_0 = \pm 90°$, $\alpha_0 = \alpha_p + \arg(0, 0)$.
6. If $\delta_p = +90°$:
   - $\delta = \theta$.

     Proof: substituting $\delta_p = 90°$ into Eq. (1δ) (or Eq. (2θ)) reduces it to $\sin\delta = \sin\theta$, whence $\delta = \theta$.
   - $\delta_0 = \theta_0$.

     Proof: a special case of the above.
   - $\alpha = \alpha_p + \phi - \phi_p - 180° \quad \ldots\delta = \theta \neq \pm 90°$

     Proof: substituting $\delta_p = 90°$ into Eq. (1α) reduces it to $\alpha = \alpha_p + \arg(-\cos\theta\cos(\phi - \phi_p), -\cos\theta\sin(\phi - \phi_p))$. For $\theta \neq \pm 90°$ this becomes $\alpha = \alpha_p + \arg(\cos(\phi - \phi_p - 180°), \sin(\phi - \phi_p - 180°))$, whence $\alpha = \alpha_p + \phi - \phi_p - 180°$.
   - $\alpha_p = \alpha_0 + (\phi_p - \phi_0) - 180° \quad \ldots\delta_0 = \theta_0 \neq \pm 90°$

     Proof: a special case of the above.
   - α is indeterminate if $\delta = \theta = \pm 90°$.

     Proof: substituting $\delta_p = 90°$, $\theta = \pm 90°$ into Eq. (1α) reduces it to $\alpha = \alpha_p + \arg(0, 0)$.
   - α_p is indeterminate if $\delta_0 = \theta_0 = \pm 90°$.

     Proof: substituting $\delta_p = 90°$, $\theta_0 = \pm 90°$ into Eq. (1α) reduces it to $\alpha_0 = \alpha_p + \arg(0, 0)$.

   Comment: for $\delta_p = 90°$ the native and celestial poles coincide.
7. If $\delta_p = -90°$:
   - $\delta = -\theta$.

     Proof: substituting $\delta_p = -90°$ into Eq. (1δ) (or Eq. (2θ)) reduces it to $\sin\delta = -\sin\theta$, whence $\delta = -\theta$.
   - $\delta_0 = -\theta_0$.

     Proof: a special case of the above.

- $\alpha = \alpha_p - (\phi - \phi_p) \quad \ldots\delta = -\theta \neq \pm 90°$

  Proof: substituting $\delta_p = -90°$ into Eq. (1α) reduces it to $\alpha = \alpha_p + \arg(\cos\theta\cos(\phi - \phi_p), -\cos\theta\sin(\phi - \phi_p))$. For $\theta \neq \pm 90°$ this becomes $\alpha = \alpha_p + \arg(\cos(\phi_p - \phi°), \sin(\phi_p - \phi))$, whence $\alpha = \alpha_p - (\phi - \phi_p)$.
- $\alpha_p = \alpha_0 - (\phi_p - \phi_0) \quad \ldots\delta_0 = -\theta_0 \neq \pm 90°$

  Proof: a special case of the above.
- α is indeterminate if $\delta = -\theta = \pm 90°$.

  Proof: substituting $\delta_p = -90°$, $\theta = \pm 90°$ into Eq. (1α) reduces it to $\alpha = \alpha_p + \arg(0, 0)$.
- α_p is indeterminate if $\delta_0 = -\theta_0 = \pm 90°$.

  Proof: substituting $\delta_p = -90°$, $\theta_0 = \pm 90°$ into Eq. (1α) reduces it to $\alpha_0 = \alpha_p + \arg(0, 0)$.

Comment: for $\delta_p = -90°$ the native pole coincides with the celestial south pole.

These results are summarized in Table 1 which also demonstrates completeness and self-consistency for values of θ_0, δ_0, and δ_p of $\pm 90°$. Indeterminate values of α_p occur for $\delta_0 = \pm 90°$; for these we *define* $\alpha_p \equiv \alpha_0$, as shown in the table. This definition is appropriate for $\theta_0 = +90°$. Other special values of θ_0 and δ_0 are

8. If $\theta_0 = \delta_0 = 0$:
   - $\alpha_p - \alpha_0 = -(\phi_p - \phi_0) = \pm 90° \quad \ldots\delta_p \neq \pm 90°$.

     Proof: substituting $\theta_0 = \delta_0 = 0$ into Eq. (2θ) gives $\cos\delta_p \cos(\alpha_0 - \alpha_p) = 0$. For $\delta_p \neq \pm 90°$ this reduces to $\cos(\alpha_0 - \alpha_p) = 0$ whence $\alpha_0 - \alpha_p = \pm 90°$.

     Now, substituting $\theta_0 = \delta_0 = 0$ and $\alpha_p - \alpha_0 = +90°$ into Eq. (2ϕ) gives $\phi_0 = \phi_p + \arg(0, -1)$ whence $\phi_0 - \phi_p = -90°$.

     Likewise, substituting $\theta_0 = \delta_0 = 0$ and $\alpha_p - \alpha_0 = -90°$ into Eq. (2ϕ) gives $\phi_0 = \phi_p + \arg(0, +1)$ whence $\phi_0 - \phi_p = +90°$.
   - δ_p is indeterminate.

     Proof: substituting $\theta_0 = \delta_0 = 0$ and $\phi_p - \phi_0 = \pm 90°$ ($\delta_p \neq \pm 90°$) into Eq. (1δ) gives $\cos\delta_p = 0/0$.

   Comment: $\phi_p = \phi_0 \pm 90°$ is required when $\theta_0 = \delta_0 = 0$; if it is given (via `LONPOLE`a) as some other value then the FITS WCS header is invalid. Also, δ_p is completely determined by `LATPOLE`a when $\theta_0 = \delta_0 = 0$.

These results are also included in the bottom part of Table 1.

All of the results in Table 1 follow from Eqs. (8), (9), and (10) of WCS Paper II subject to the conditions (1-6) listed after the equations where it is understood that these are to be considered in sequence.

2. AZP conic sections

Derivation of the equations of the conic sections for the projected parallels of native latitude for the `AZP` projection.

Projection equations for zenithal perspective projection are:

$$x = R\sin\phi\,, \tag{3}$$
$$y = -R\sec\gamma\cos\phi\,, \tag{4}$$

where

$$R = \frac{180°}{\pi}\frac{(\mu+1)\cos\theta}{(\mu+\sin\theta)+\cos\theta\cos\phi\tan\gamma}\,. \tag{5}$$

For constant θ, each parallel of native latitude defines a cone with apex at the point of projection. This cone intersects the tilted plane of projection in a conic section. Write

$$Y = -y\cos\gamma\,, \tag{6}$$
$$R = \frac{1}{\kappa + \lambda\cos\phi}\,, \tag{7}$$

where

$$\kappa = \frac{\pi(\mu+\sin\theta)}{180(\mu+1)\cos\theta}\,, \tag{8}$$
$$\lambda = \frac{\pi\tan\gamma}{180(\mu+1)}\,, \tag{9}$$

so that Eqs. (3) and (4) become

$$x = \frac{\sin\phi}{\kappa+\lambda\cos\phi}\,, \tag{10}$$
$$Y = \frac{\cos\phi}{\kappa+\lambda\cos\phi}\,. \tag{11}$$

Combining Eqs. (10) and (11) we have

$$x^2 + Y^2 = \frac{1}{(\kappa+\lambda\cos\phi)^2}\,, \tag{12}$$

but Eq. (11) gives

$$\cos\phi = \frac{\kappa Y}{1-\lambda Y}\,, \tag{13}$$

whence

$$(x^2+Y^2)(\frac{\kappa}{1-\lambda Y})^2 = 1\,, \tag{14}$$

$$\kappa^2 x^2 + (\kappa^2 - \lambda^2)Y^2 + 2\lambda Y = 1\,. \tag{15}$$

Second order equations of this general form are those of a conic section. The quantity

$$C = \kappa^2 - \lambda^2\,, \tag{16}$$
$$= \frac{\pi^2\left[(\mu+\sin\theta)^2 - \tan^2\gamma\cos^2\theta\right]}{180^2(\mu+1)^2\cos^2\theta}\,, \tag{17}$$

determines the nature of the curve:

$$\begin{array}{lcl} C > 0: & \text{...ellipse,} \\ C = 0: & \text{...parabola,} \\ C < 0: & \text{...hyperbola.} \end{array} \tag{18}$$

The condition $C = 0$ is satisfied when

$$\theta = \gamma - \sin^{-1}(\mu\cos\gamma)\,. \tag{19}$$

Completing the square in Eq. (15) gives, for $C \neq 0$,

$$\kappa^2 x^2 + C(Y + \frac{\lambda}{C})^2 = \frac{\kappa^2}{C} \tag{20}$$

whence for $C > 0$

$$\frac{x^2}{a^2} + \frac{(y - y_0)^2}{b^2} = 1 \,, \tag{21}$$

where

$$a = \frac{1}{\sqrt{C}} \,, \tag{22}$$

$$b = \frac{\kappa}{C \cos\gamma} \,, \tag{23}$$

$$y_0 = \frac{\lambda}{C} \,. \tag{24}$$

Since a, b and y_0 are functions of θ the eccentricity of the projected parallels varies as does the offset of their centres in y.