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Introduction to the Fourier Transform

Robert Braun with materials from:

John Dickey & Bob Watson (UTas) and John Braher (UNM)

Statement of Fourier's Theorem

Suitable periodic functions, with a period of 2π, may be represented as

$$f(\theta) =$$

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\theta + b_n \sin n\theta\right)$$

(called a **Fourier series**)

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Here

$$a_0 = \frac{1}{\pi}\int_{\theta_0}^{\theta_0+2\pi} f(\theta)\, d\theta$$

$$a_n = \frac{1}{\pi}\int_{\theta_0}^{\theta_0+2\pi} f(\theta)\cos(n\theta)\, d\theta$$

$$b_n = \frac{1}{\pi}\int_{\theta_0}^{\theta_0+2\pi} f(\theta)\sin(n\theta)\, d\theta$$

The word **"suitable"** covers most functions you meet in physics

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The othorgonality relations:

$$\int_{\theta_0}^{\theta_0+2\pi} \cos m\theta \cos n\theta \, d\theta = 0 \quad m \neq n$$

$$= \pi \quad m = n$$

$$\int_{\theta_0}^{\theta_0+2\pi} \sin m\theta \sin n\theta \, d\theta = 0 \quad m \neq n$$

$$= \pi \quad m = n$$

$$\int_{\theta_0}^{\theta_0+2\pi} \sin m\theta \cos n\theta \, d\theta = 0 \quad (\text{all } m, n)$$

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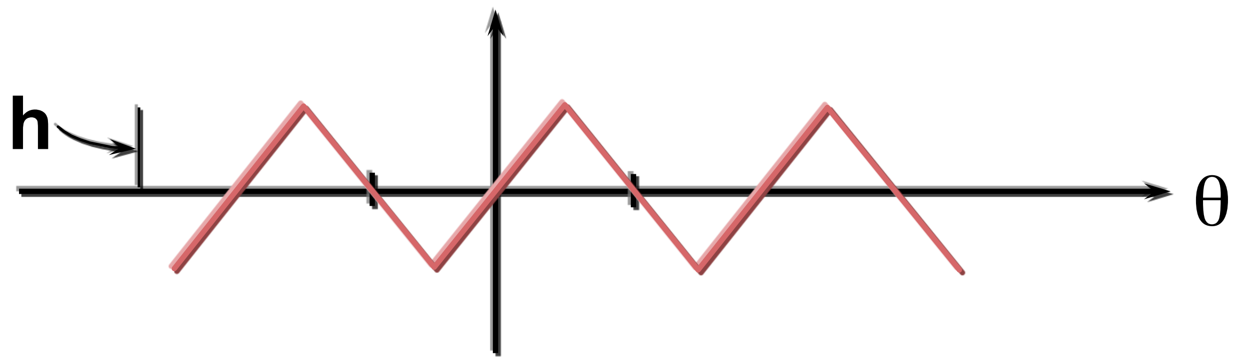
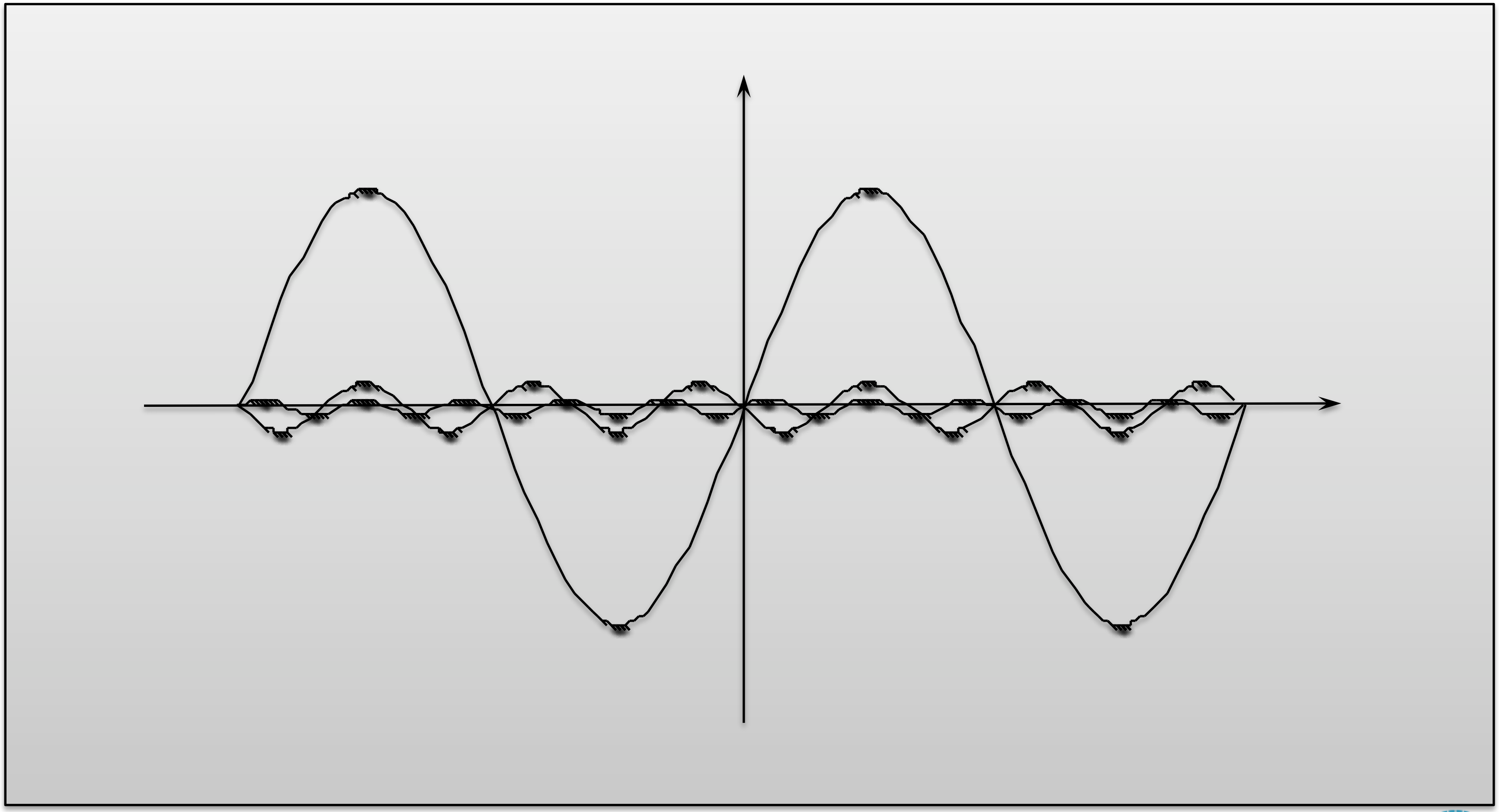
Triangular wave



$$f(\theta) = \left(\frac{8h}{\pi^2}\right)\left(\frac{\sin\theta}{1^2} - \frac{\sin 3\theta}{3^2} + \frac{\sin 5\theta}{5^2} - \frac{\sin 7\theta}{7^2} \ldots\right)$$

Triangular wave

First 3 series terms:



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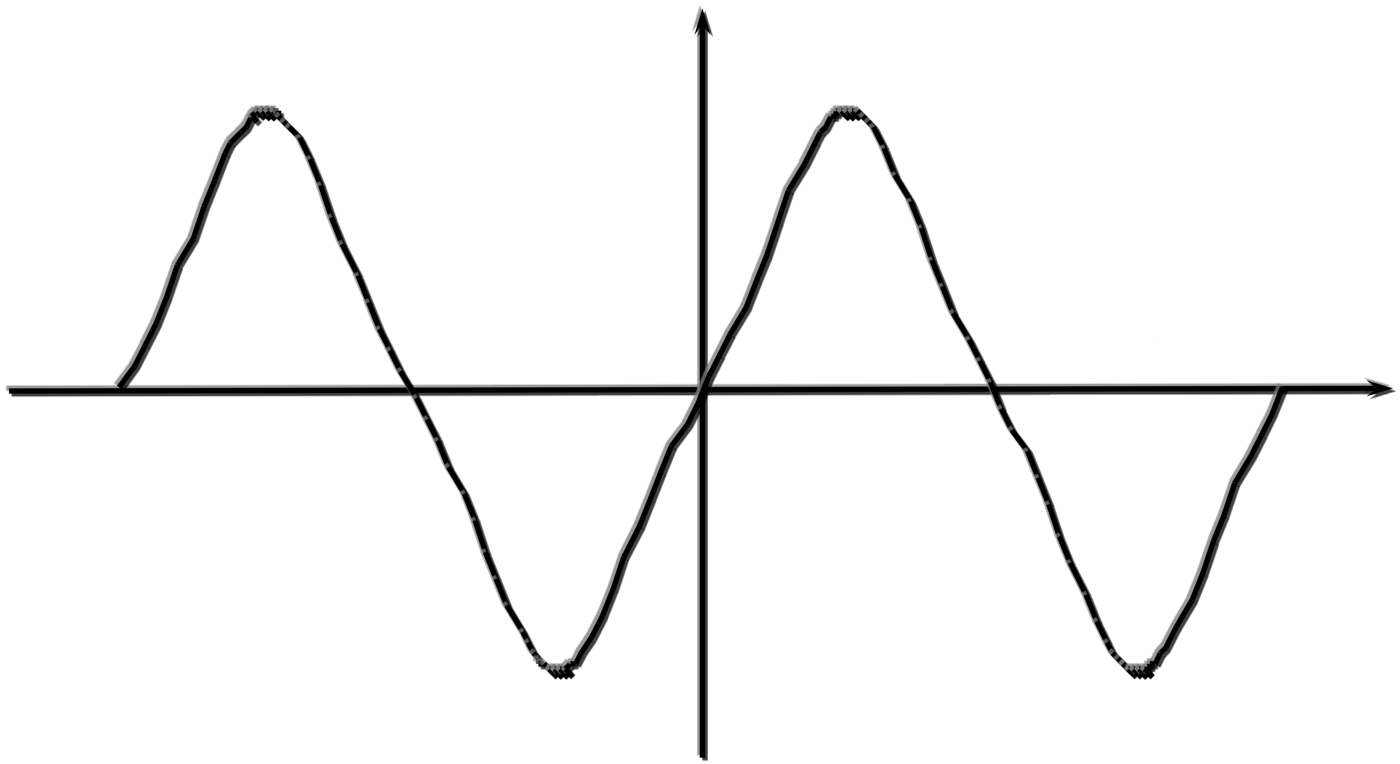
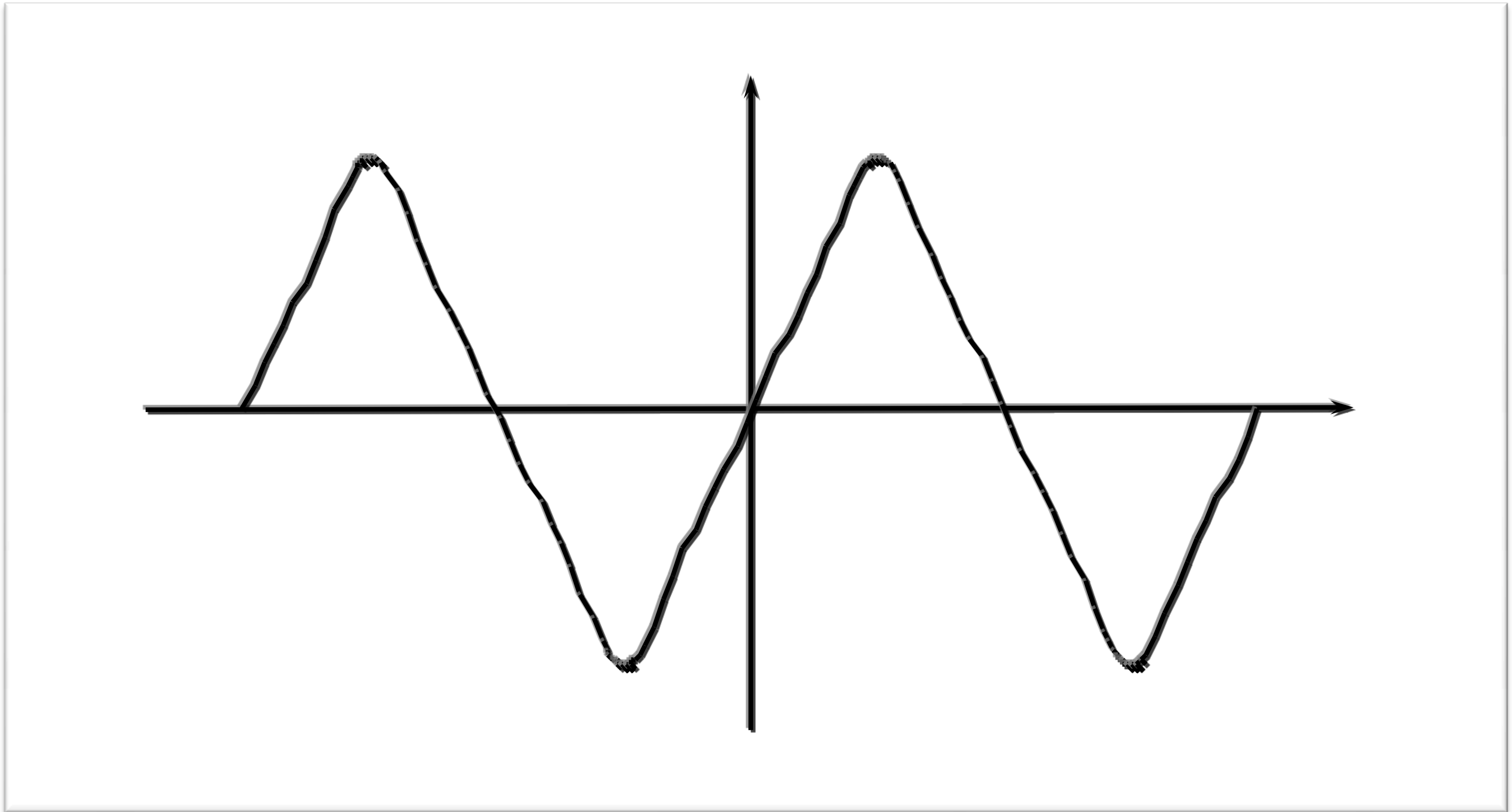
Triangular wave

Sum of first 2 terms:



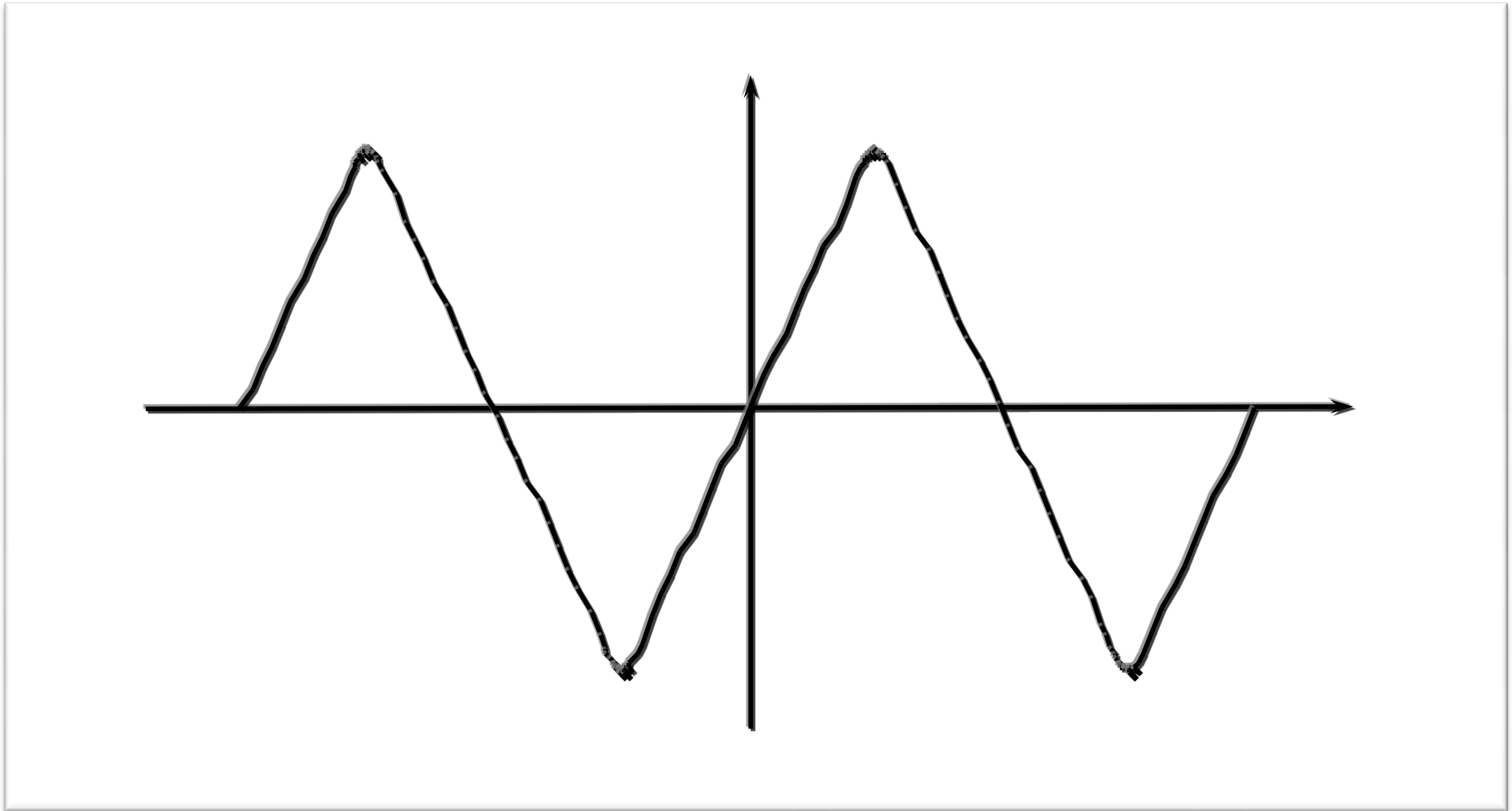
Triangular wave

Sum of first 3 terms:



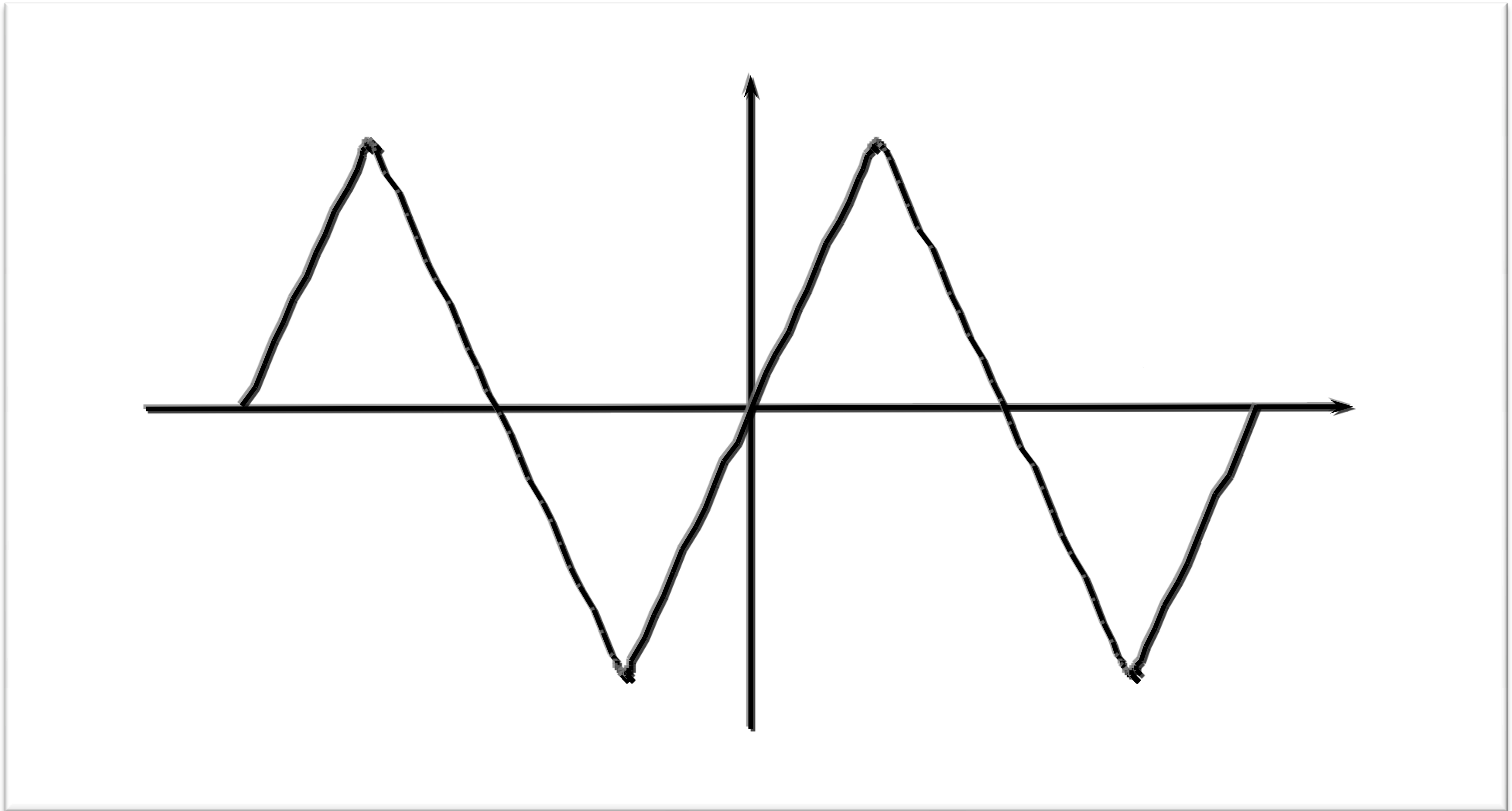
Triangular wave

Sum of first 5 terms:



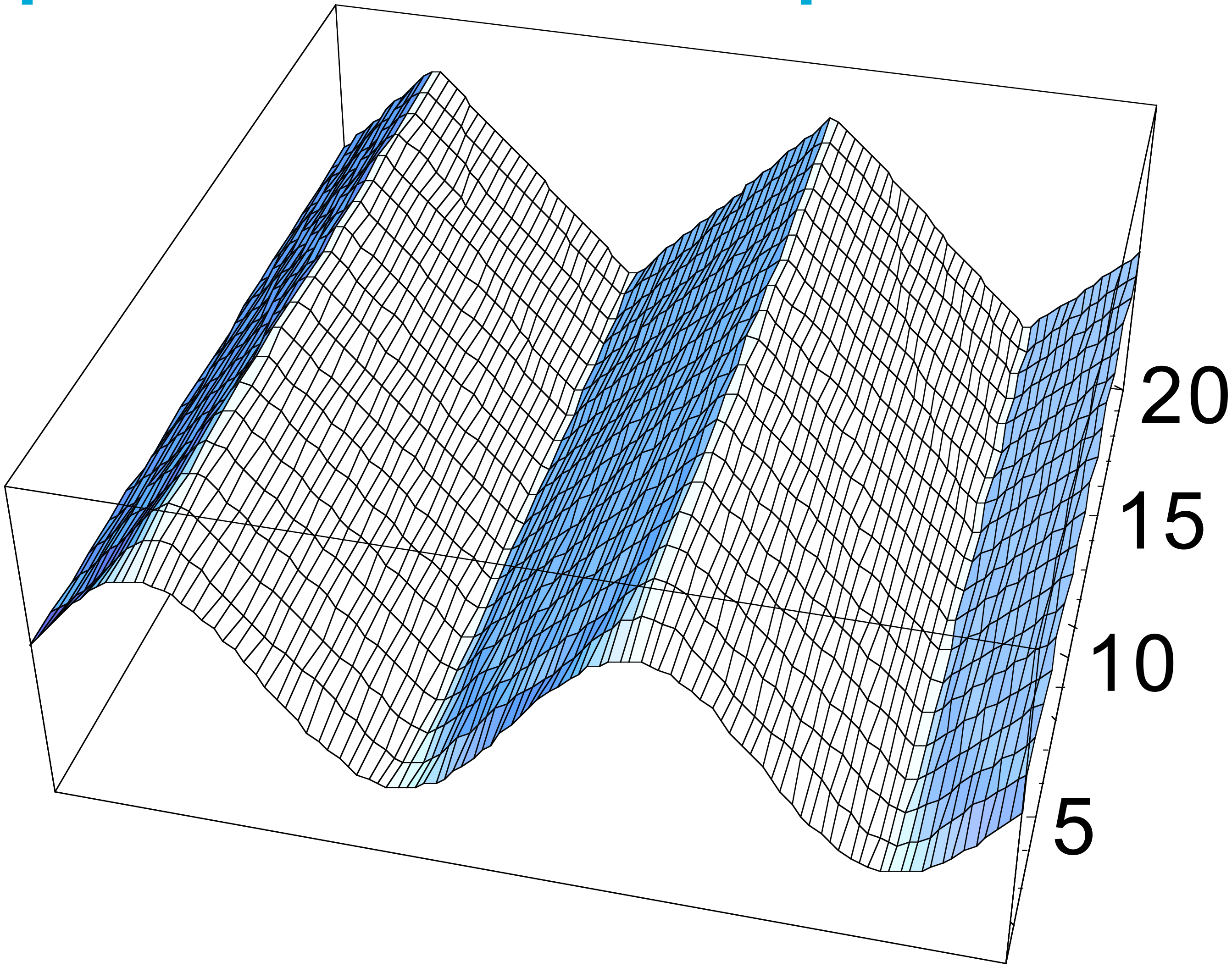
Triangular wave

Sum of first 10 terms:



Triangular wave

All partial series sums up to 20 terms:



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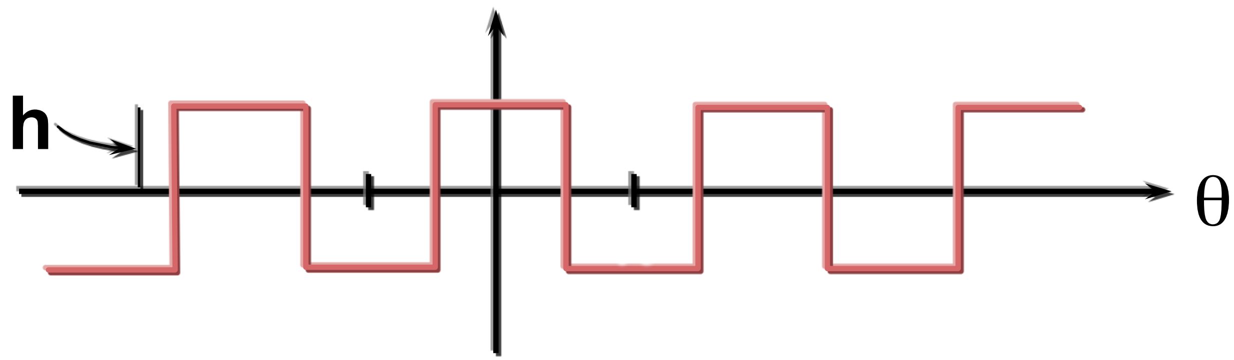
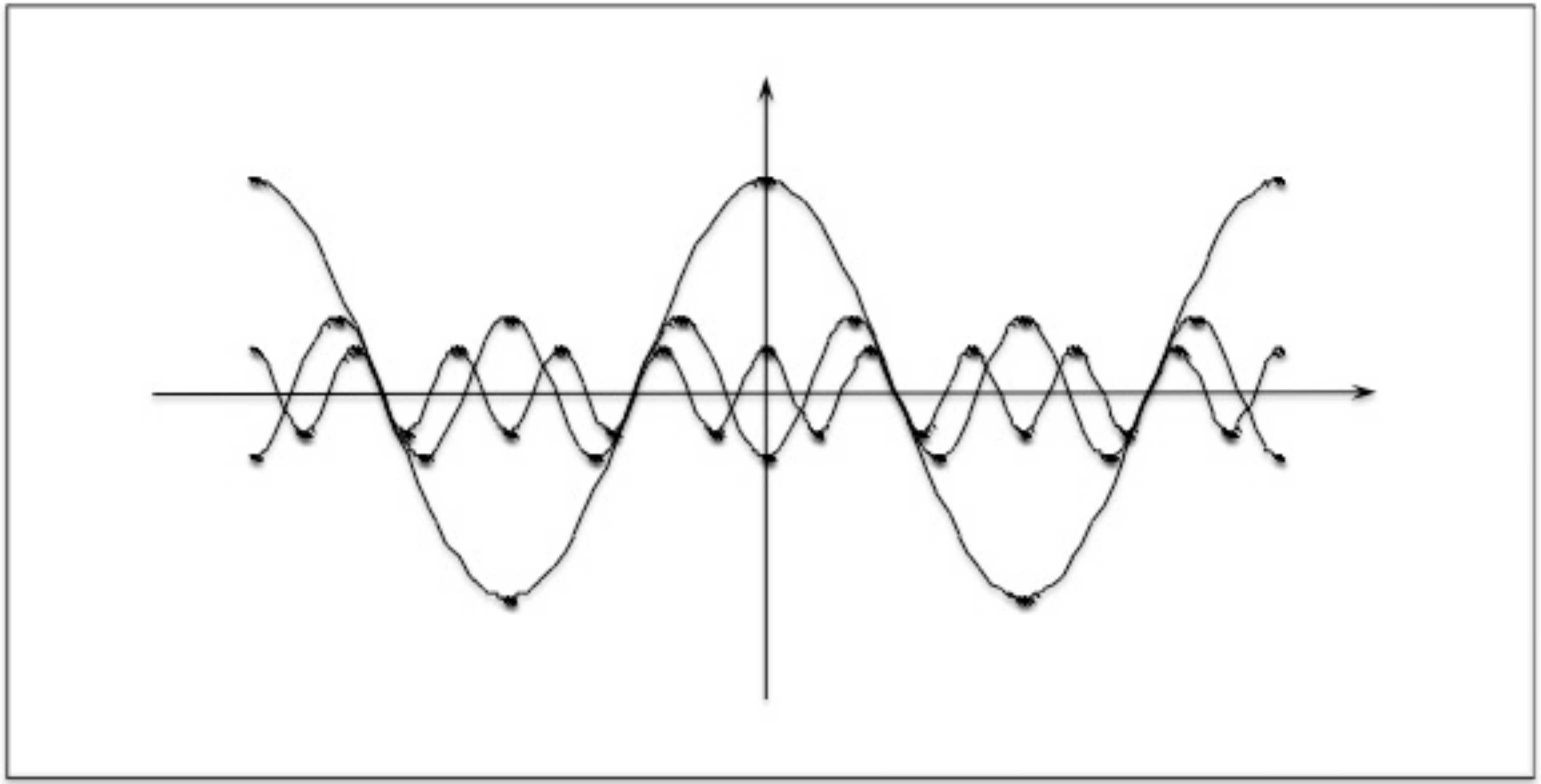
Even square wave



$$f(\theta) = \left(\frac{4h}{\pi}\right)\left(\cos\theta - \tfrac{1}{3}\cos 3\theta \;\ldots\; + \tfrac{1}{5}\cos 5\theta - \tfrac{1}{7}\cos 7\theta \;\ldots\right)$$

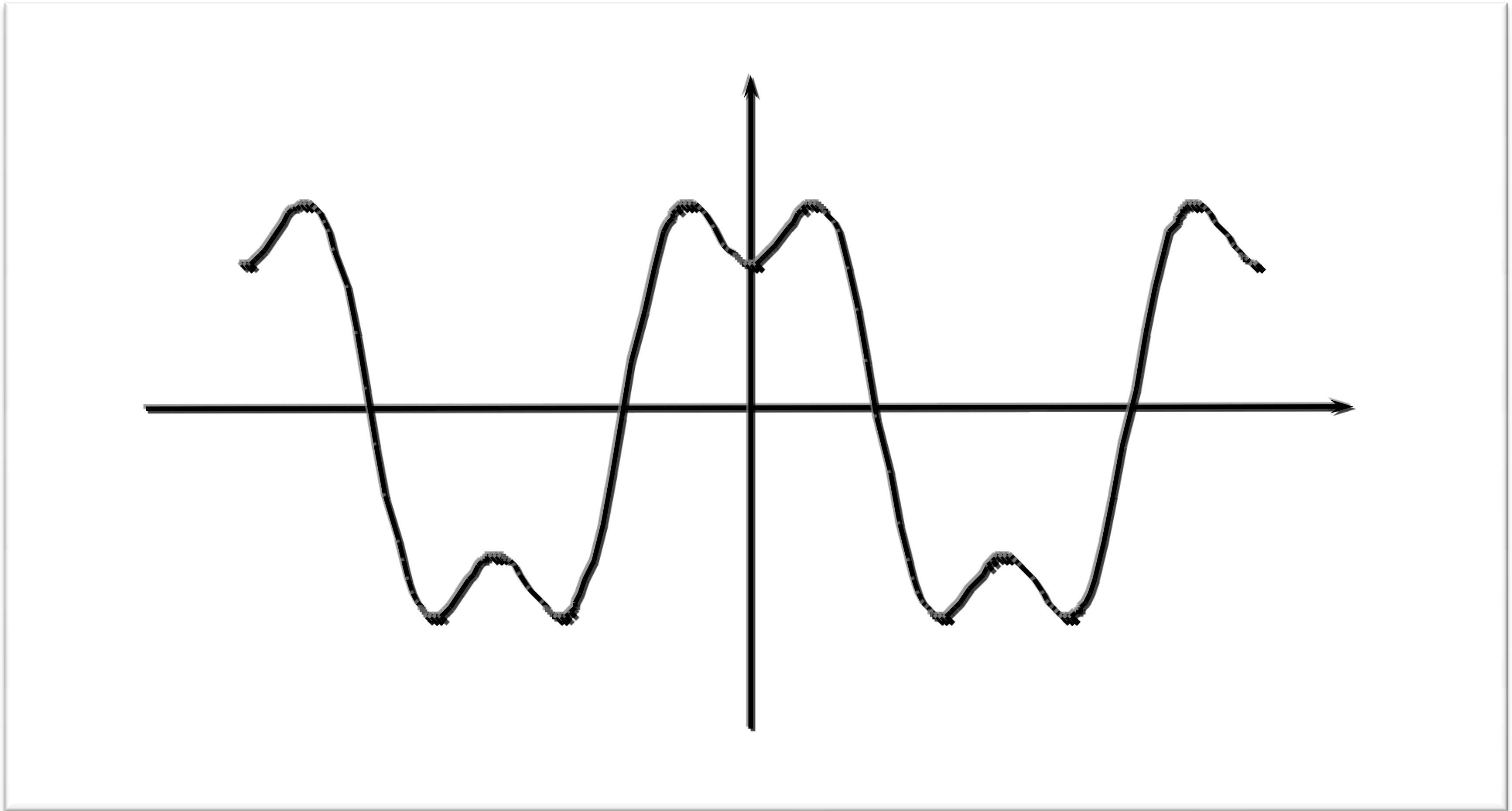
Even square wave

- First three terms



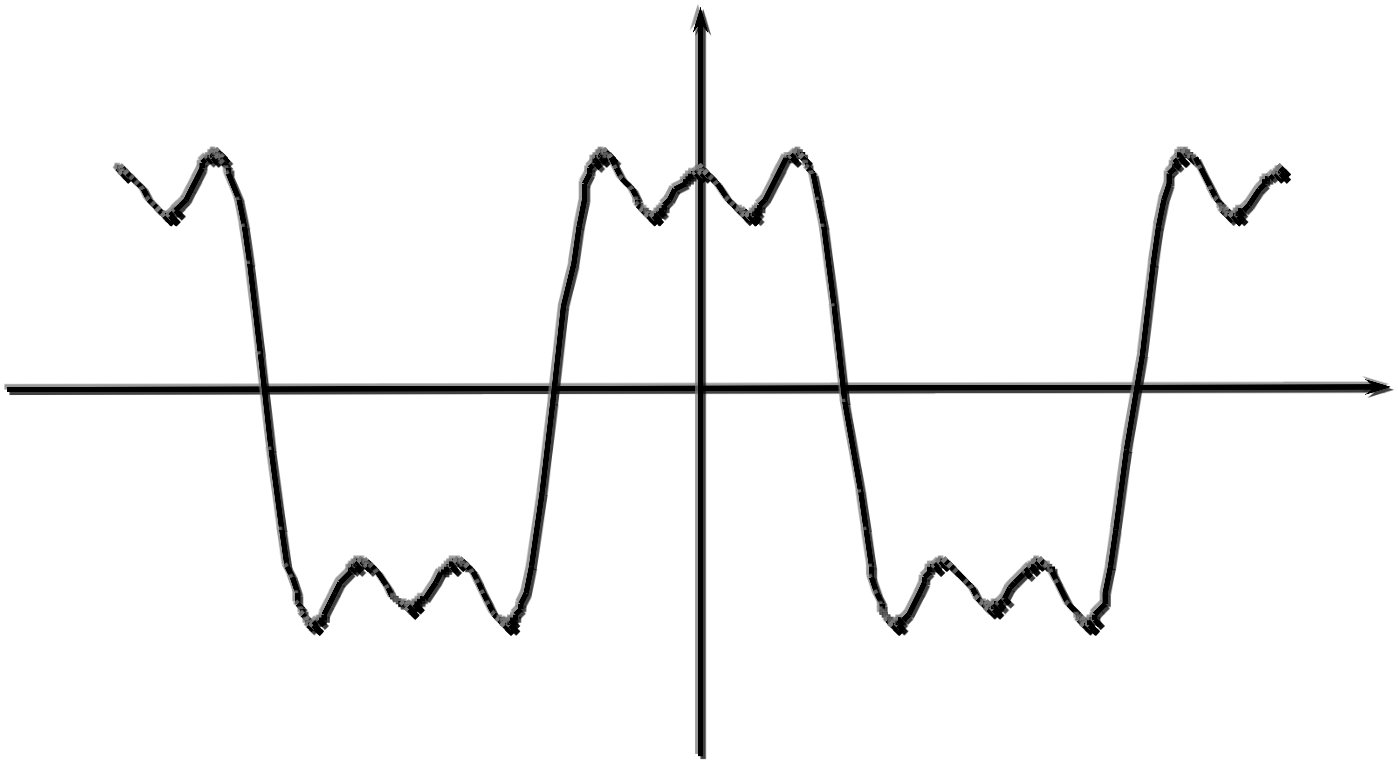
Even square wave

- Sum of first two terms



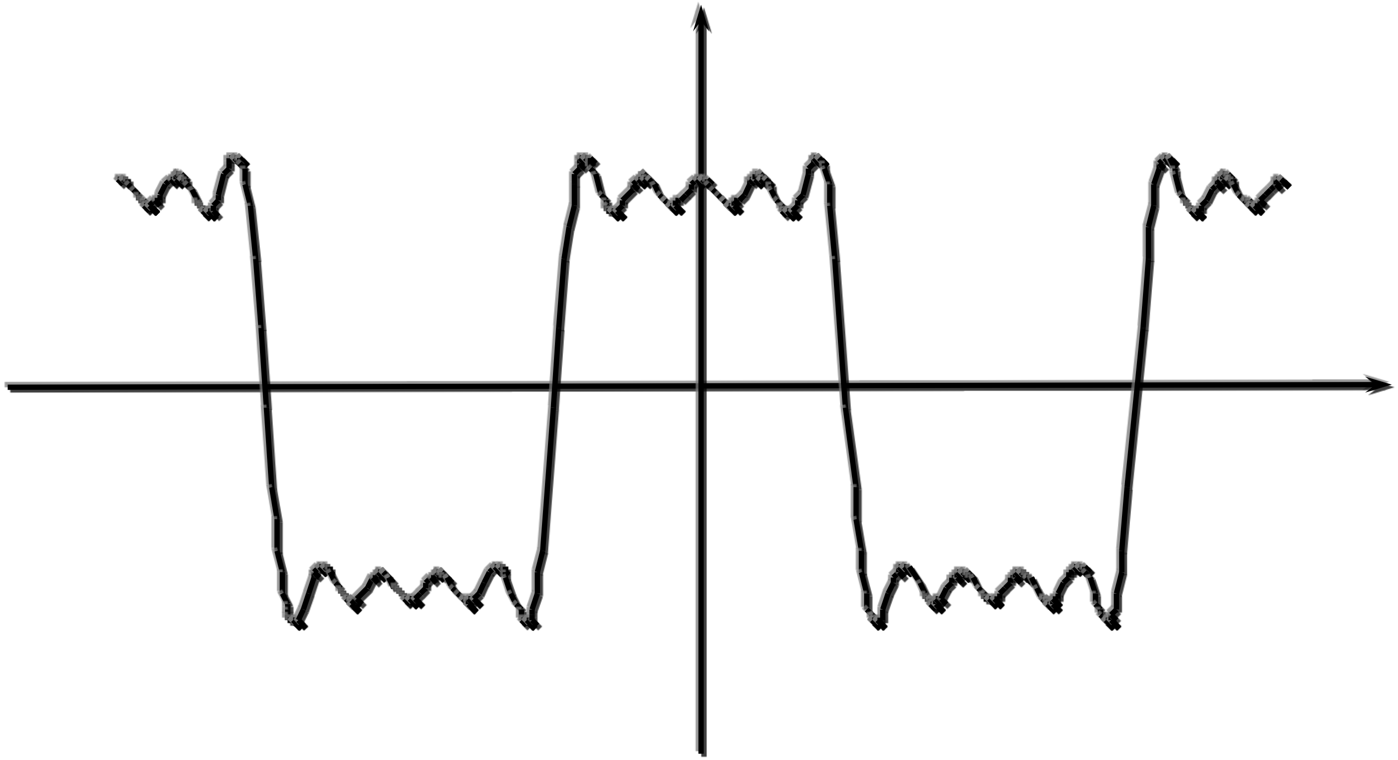
Even square wave

- Sum of first three terms



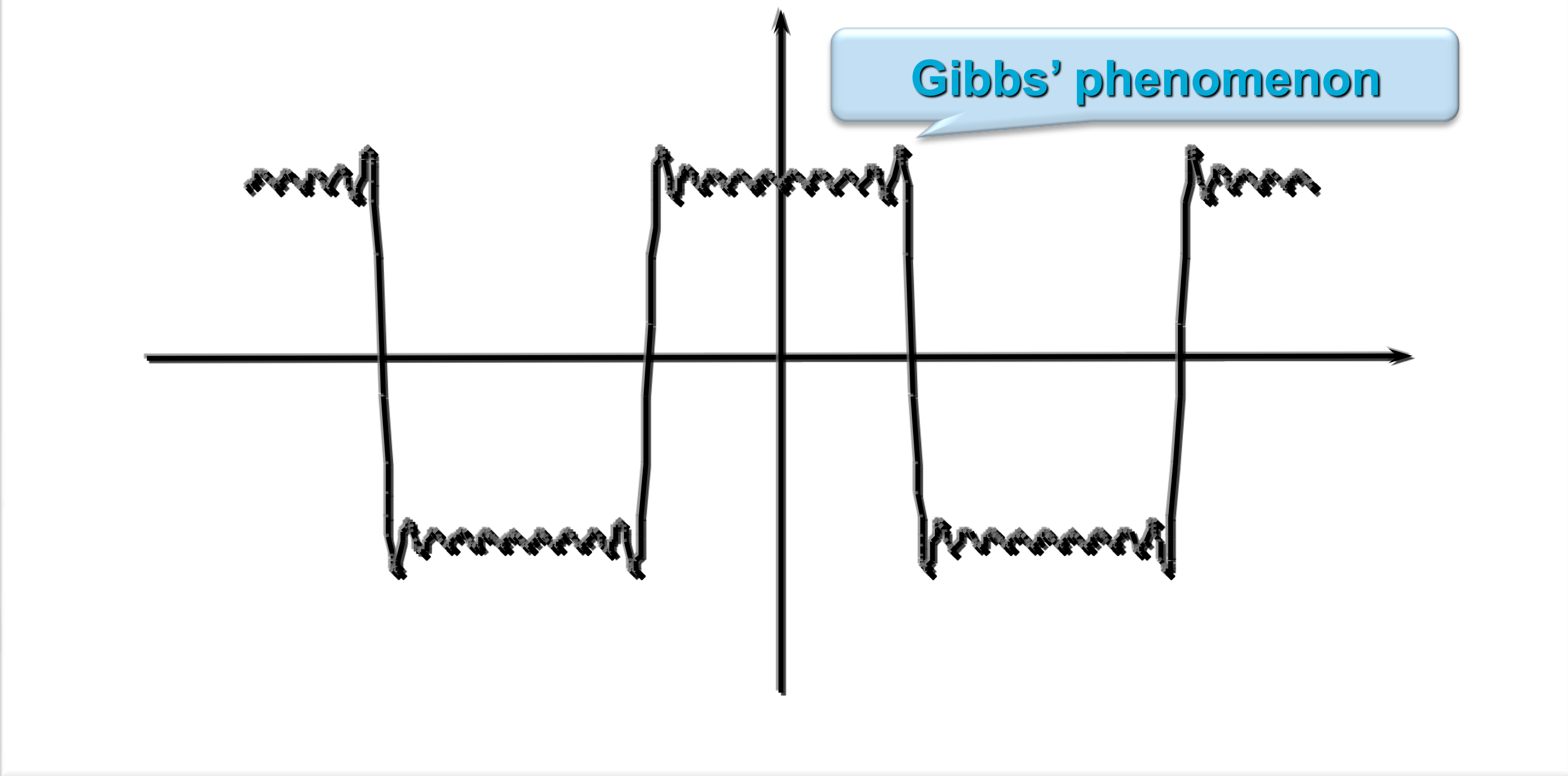
Even square wave

- Sum of first five terms



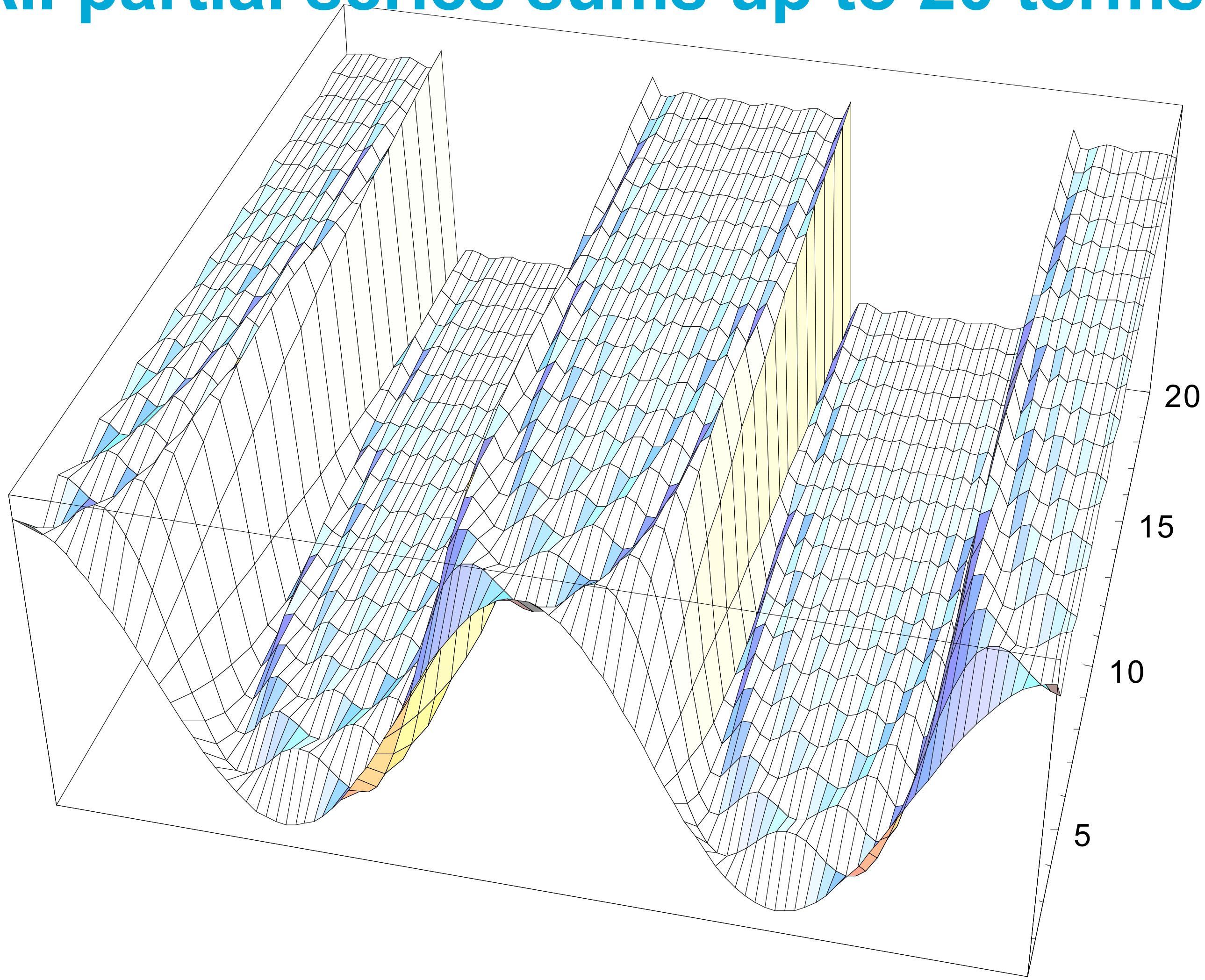
Even square wave

- Sum of first ten terms



Even square wave

- **All partial series sums up to 20 terms:**



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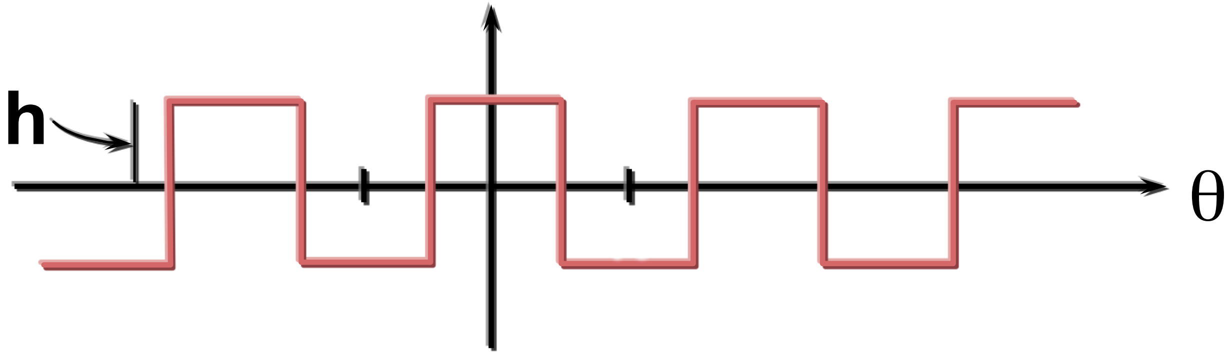
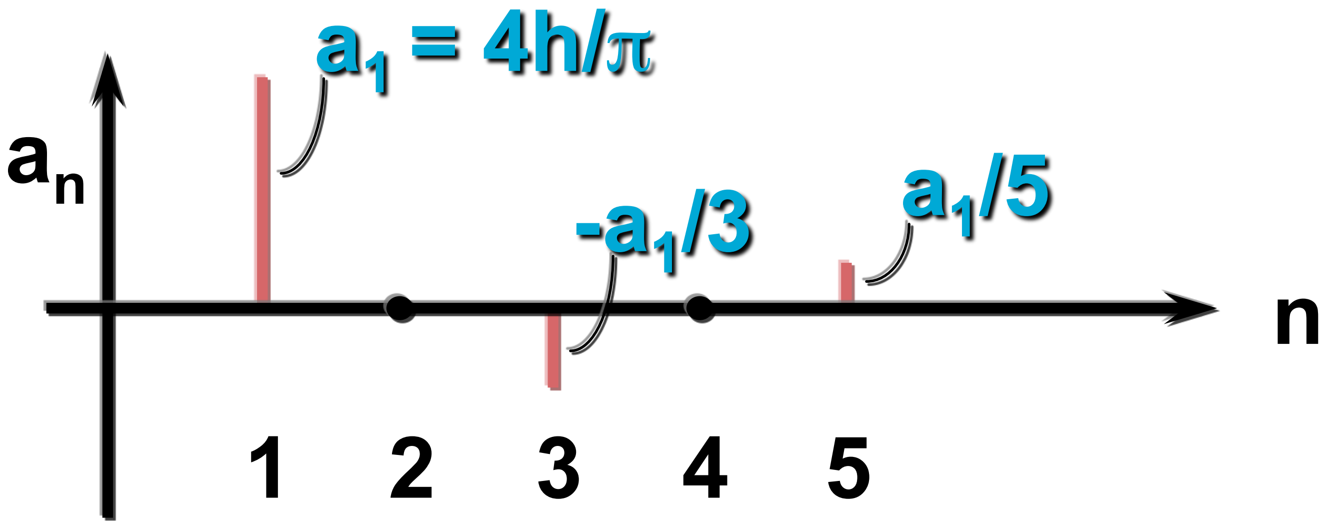
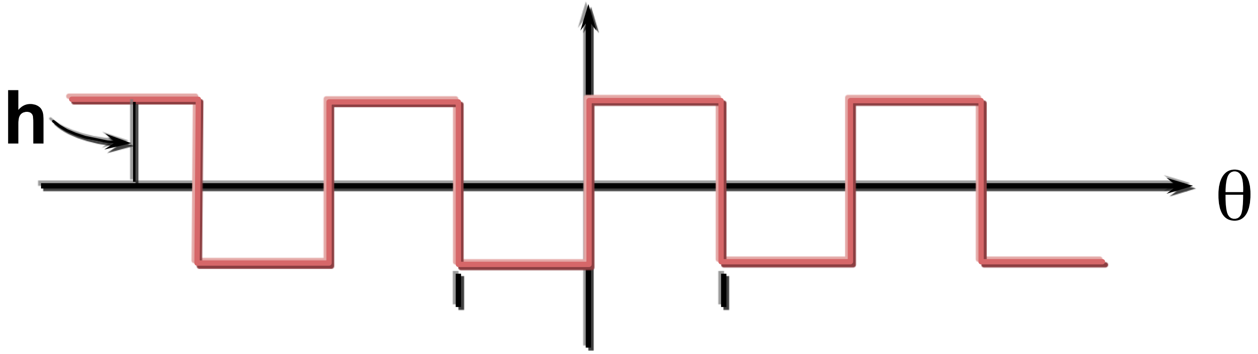
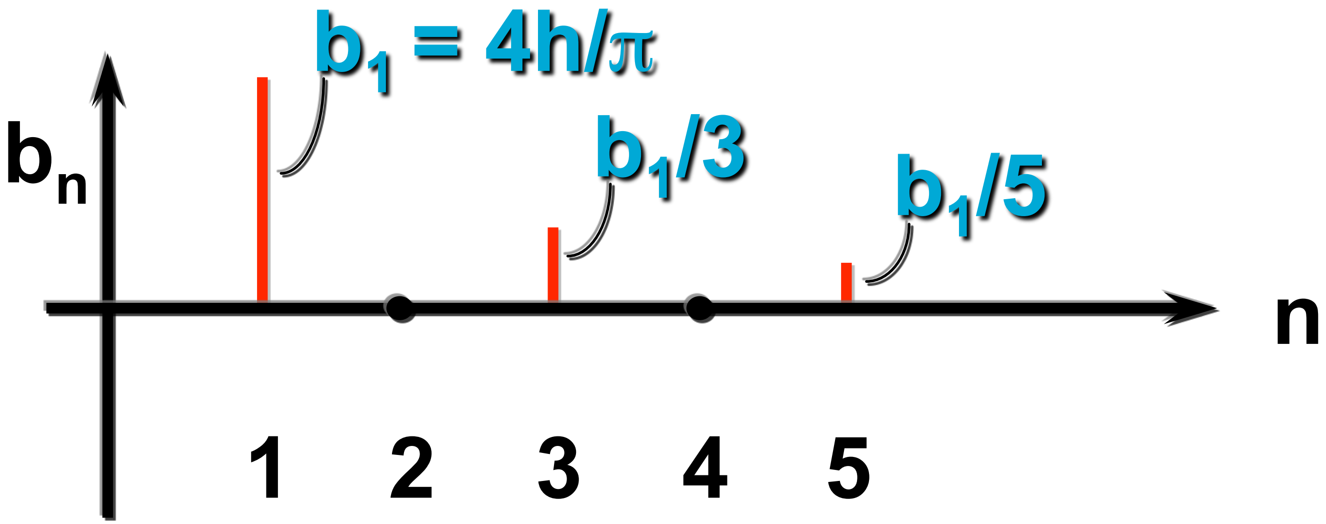
The Fourier Series Spectrum: Even square wave





The Fourier Series Spectrum: Odd square wave





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Complex form of Fourier's Theorem

Suitable periodic functions, with a period of 2π, may be represented as

$$f(\theta) = \sum_{n=-\infty}^{+\infty} \overline{C}_n e^{jn\theta}$$

where

$$\overline{C}_n = \frac{1}{2\pi}\int_{\theta_0}^{\theta_0+2\pi} f(\theta)e^{-jn\theta}\, d\theta$$

Temporally periodic case

If we have a function $f(t)$, with temporal period T, which we wish to express as a Fourier series, then change our $f(\theta)$ series using

$$t / T = \theta / 2\pi$$

$$\text{i.e. } \theta = \left(\frac{2\pi}{T}\right)t$$

$$\delta\theta = \left(\frac{2\pi}{T}\right)\delta t$$

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So the Fourier series for f(t) is

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\tfrac{2\pi n t}{T}\right) + b_n \sin\left(\tfrac{2\pi n t}{T}\right)\right]$$

where

$$a_0 = \tfrac{2}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

$$a_n = \tfrac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos\left(\tfrac{2\pi n t}{T}\right)\,dt$$

$$b_n = \tfrac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin\left(\tfrac{2\pi n t}{T}\right)\,dt$$

Temporally periodic case

T is called the fundamental period of $f(t)$ and $\omega_1 = 2\pi/T$ is fundamental angular frequency

In terms of ω_1: f(t) =

$$\tfrac{1}{2}a_0 + \sum_{n=1}^{\infty}\left[a_n\cos(n\omega_1 t) + b_n\sin(n\omega_1 t)\right]$$

where

$$a_0 = \tfrac{2}{T}\int_{t_0}^{t_0+T} f(t)\,dt$$

$$a_n = \tfrac{2}{T}\int_{t_0}^{t_0+T} f(t)\cos(n\omega_1 t)\,dt$$

$$b_n = \tfrac{2}{T}\int_{t_0}^{t_0+T} f(t)\sin(n\omega_1 t)\,dt$$

Temporally periodic case

The corresponding complex form of f(t) in terms of ω_1 is

$$f(t) = \sum_{n=-\infty}^{+\infty} \overline{C}_n e^{jn\omega_1 t}$$

where

$$\overline{C}_n = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)\, e^{-jn\omega_1 t}\, dt$$

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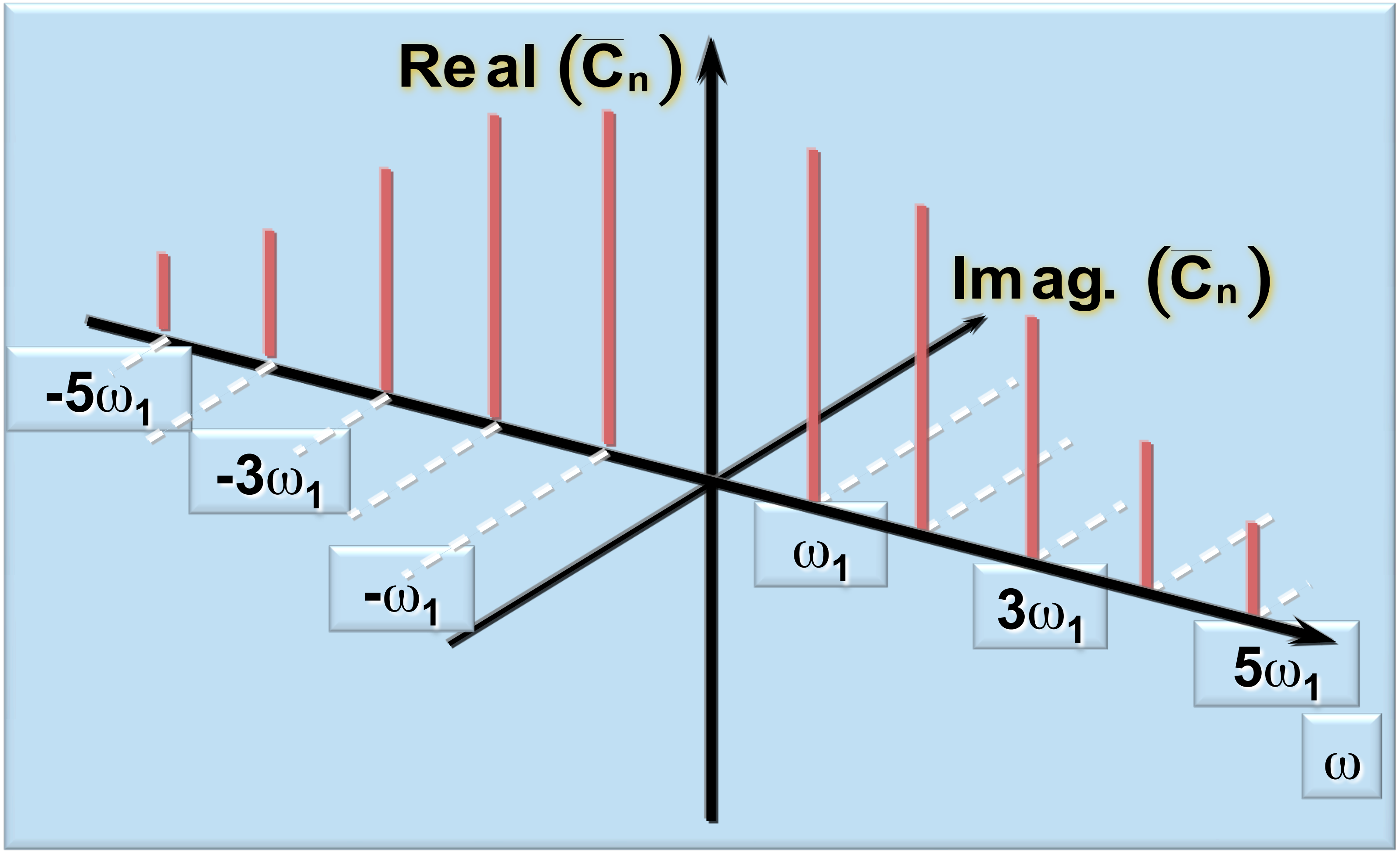
Visualizing the complex C_n



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From Fourier Series to Fourier Transform

$$f(t) = \sum_{n=-\infty}^{+\infty} \overline{C}_n e^{jn\omega_1 t}$$

$$= \sum_{n=-\infty}^{+\infty} \left(\frac{1}{T} \int_{-T/2}^{+T/2} f(t) e^{-jn\omega_1 t} \, dt \right) e^{jn\omega_1 t}$$

and substitute

$$\begin{cases} \omega = n\omega_1 \\ \dfrac{\delta\omega}{\pi} = \dfrac{2}{T} \end{cases}$$

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From Fourier Series to Fourier Transform

Hence

$$f(t) = \sum_{\omega=-\infty}^{+\infty} \left(\frac{\delta\omega}{2\pi} \int_{-T/2}^{+T/2} f(t) e^{-j\omega t}\, dt \right) e^{j\omega t}$$

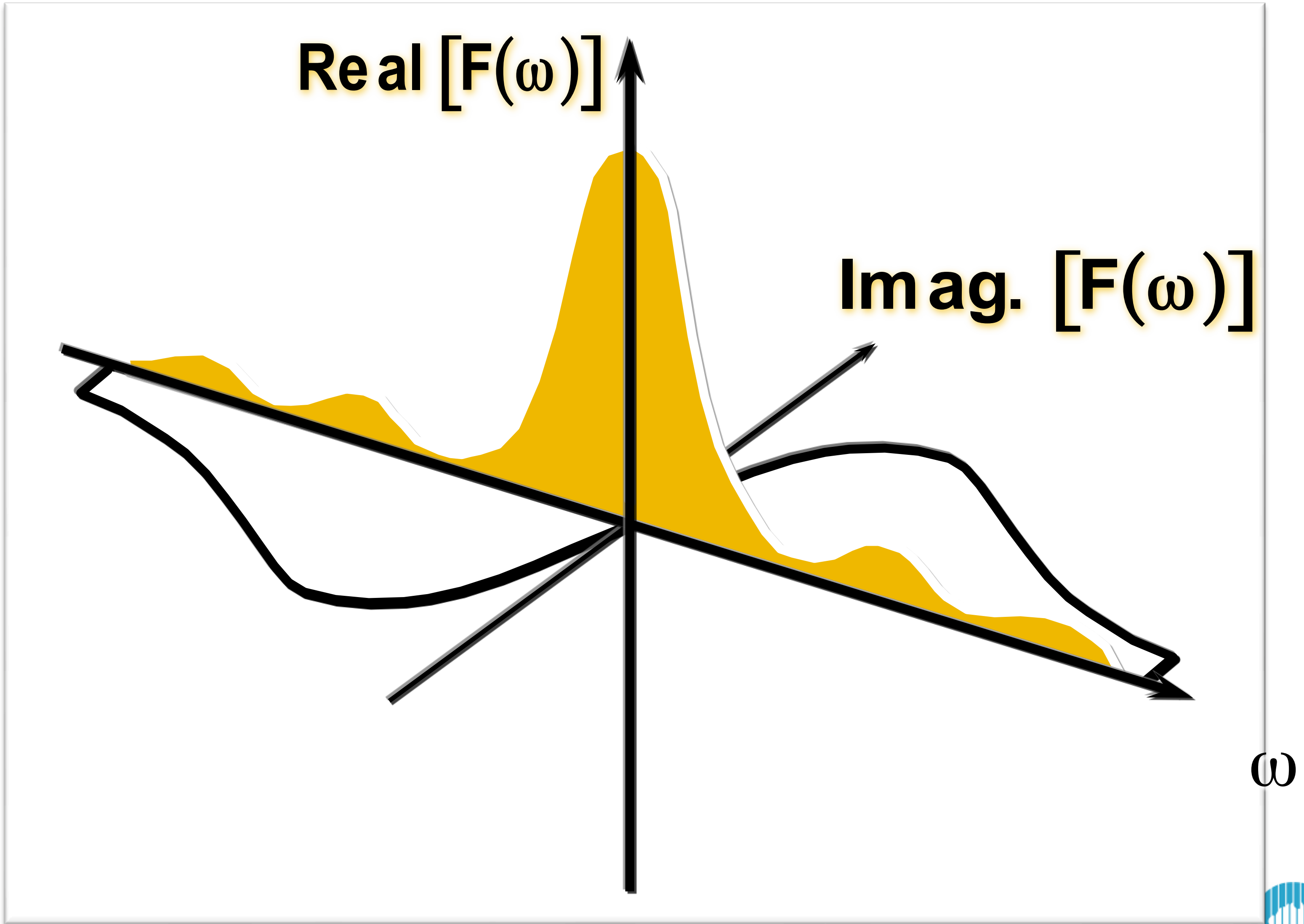
As $T \rightarrow \infty$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t}\, d\omega$$

where

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) e^{-j\omega t}\, dt$$

Visualizing the complex F(ω)



Temporally periodic case

Replace ω by the frequency $\nu = \omega/2\pi$

Thus we see,

$$f(t) = \int_{-\infty}^{+\infty} F(\nu)e^{j2\pi\nu t}\, d\nu$$

$$F(\nu) = \int_{-\infty}^{+\infty} f(t)e^{-j2\pi\nu t}\, dt$$

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Spatially periodic case

Or in terms of a spatial coordinate ***x*** and the corresponding spatial frequency ***u***, or any other coordinate pair that are dimensionless in the product!

$$f(x) = \int_{-\infty}^{+\infty} F(u) e^{j2\pi ux}\, du$$

$$F(u) = \int_{-\infty}^{+\infty} f(x) e^{-j2\pi ux}\, dx$$

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Pitfalls of discrete and finite sampling

- Aliasing: A sampling rate of twice as fine as the smallest fluctuations must be used to allow accurate representation of arbitrary forms
- Leakage: A non-periodic function can only be represented accurately by an FT if one artificially insures the required boundary conditions, for example by zero-padding (see examples below)

Two Dimensional Fourier Transforms

For a function of two (or more) dimensions, the Fourier Transform is done by transforming in each dimension separately.

$$f(x,y) \;=\; \int\int_{-\infty}^{+\infty} F(u,v)\; e^{-2\pi j(ux+vy)}\; \mathrm{d}u\; \mathrm{d}v$$

$$= \int_{-\infty}^{+\infty} e^{-2\pi jux} \int_{-\infty}^{+\infty} e^{-2\pi jvy}\; F(u,v)\; \mathrm{d}u\; \mathrm{d}v$$

$$F(u,v) \;=\; \int\int_{-\infty}^{+\infty} f(x,y)\; e^{2\pi j(ux+vy)}\; \mathrm{d}x\; \mathrm{d}y$$

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Properties of the Fourier Transform

Fourier transforms are sometimes of use in physics due to their direct physical interpretation (see later), but often their usefulness is more indirect

Properties of the Fourier Transform

In particular, a great usefulness comes from their help in solving awkward differential equations

Here we list several general transform properties of benefit in this and other contexts

1. Linearity

If

$$\begin{cases} f_1(t) \rightarrow F_1(\omega) \\ f_2(t) \rightarrow F_2(\omega) \end{cases}$$

then

$$\left[c_1 f_1(t) + c_2 f_2(t)\right] \rightarrow \left[c_1 F_1(\omega) + c_2 F_2(\omega)\right]$$

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2. Scale change

If

$$f(t) \rightarrow F(\omega)$$

then

$$\begin{cases} f(at) \rightarrow \frac{1}{a} F\left(\frac{\omega}{a}\right) \\ \text{or} \\ \frac{1}{a} f\left(\frac{t}{a}\right) \rightarrow F(a\omega) \end{cases}$$

Thus compressing a time function expands its spectrum in frequency and vice-versa

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3. Delay and modulation

If

$$f(t) \rightarrow F(\omega)$$

then

$$\begin{cases} f(t - t_0) \rightarrow e^{-j\omega t_0} F(\omega) \\ \text{or} \\ e^{j\omega_0 t} f(t) \rightarrow F(\omega - \omega_0) \end{cases}$$

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3. Delay and modulation

Thus :

- delaying a time function **f(t)** simply adds a linear phase **($-\omega t_0$)** to the original **F(ω)** phase

- modulating **f(t)** by **$e^{j\omega_0 t}$** shifts the spectrum by ω_0

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3. Delay and modulation

Similarly, if one modulates by the real function $\cos(\omega_o t)$ we have:

$$\cos(\omega_0 t) f(t) \rightarrow$$

$$\tfrac{1}{2}\left[F(\omega - \omega_0) + F(\omega + \omega_0)\right]$$

Thus this real modulation gives a spectrum which is the average of two copies of the original form, one shifted by $+\omega_0$ and the other by $-\omega_0$

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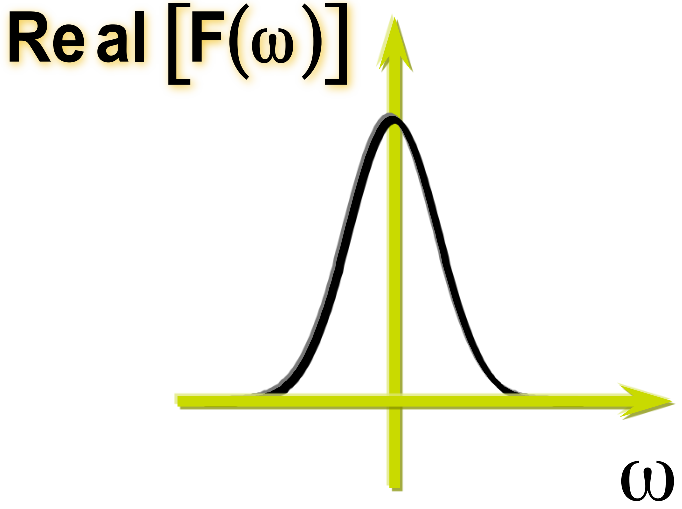
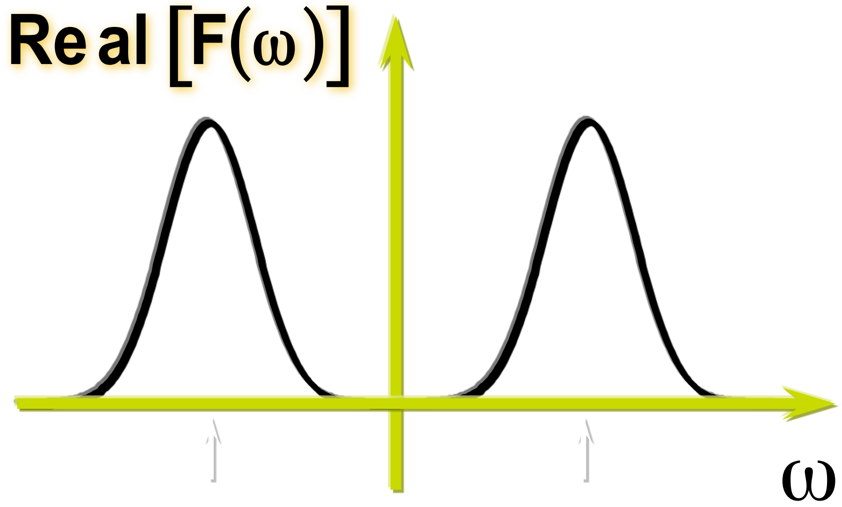
3. Delay and modulation

i.e. if f(t) transforms to



then $[\cos(\omega_o t)\, f(t)]$ transforms to



4. Differentiation and integration

If

$$f(t) \rightarrow F(\omega)$$

then

$$\begin{cases} \dfrac{df(t)}{dt} \rightarrow j\omega F(\omega) \\ \text{and} \\ \int f(t)\,dt \rightarrow \dfrac{F(\omega)}{j\omega} \end{cases}$$

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4. Differentiation and integration

- Thus taking the transform reduces differentiation and integration to the simpler operations of multiplication or division by $j\omega$ in the frequency domain. Then finding the inverse transform gives the final result in the time domain

- This is analogous to the taking of logarithms to reduce " × and ÷ " to the simpler operations of " + and - ". In that case finding the anti-log gives the final result

5. Multiplication and convolution

If

$$\begin{cases} f_1(t) \rightarrow F_1(\omega) \\ f_2(t) \rightarrow F_2(\omega) \end{cases}$$

then

$$f_1(t)f_2(t) \rightarrow \frac{1}{2\pi}\int_{-\infty}^{+\infty} F_1(\eta)F_2(\omega - \eta)d\eta$$

and

$$\int_{-\infty}^{+\infty} f_1(\eta)f_2(t - \eta)d\eta \rightarrow F_1(\omega)F_2(\omega)$$

5. Multiplication and convolution

Thus :

- a product of functions has a transform that is the convolution of the individual transforms
- conversely if you wish to convolve functions you can simply multiply their individual transforms

6. Parseval's theorem

If

$$\mathbf{f(t) \rightarrow F(\omega)}$$

then

$$\int_{-\infty}^{+\infty} \left| f(t) \right|^2 dt = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \left| F(\omega) \right|^2 d\omega$$

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Pictorial Examples

- All illustrations on 128x128 pixel grid
- Brightness > 0
- Left is a pure horizontal cosine of 8 cycles
- Right is a pure vertical cosine of 32 cycles
- Fourier Transforms (below) have origin at center, **only amplitude is shown !!**
- Note the inverse relationship between scales in the two planes

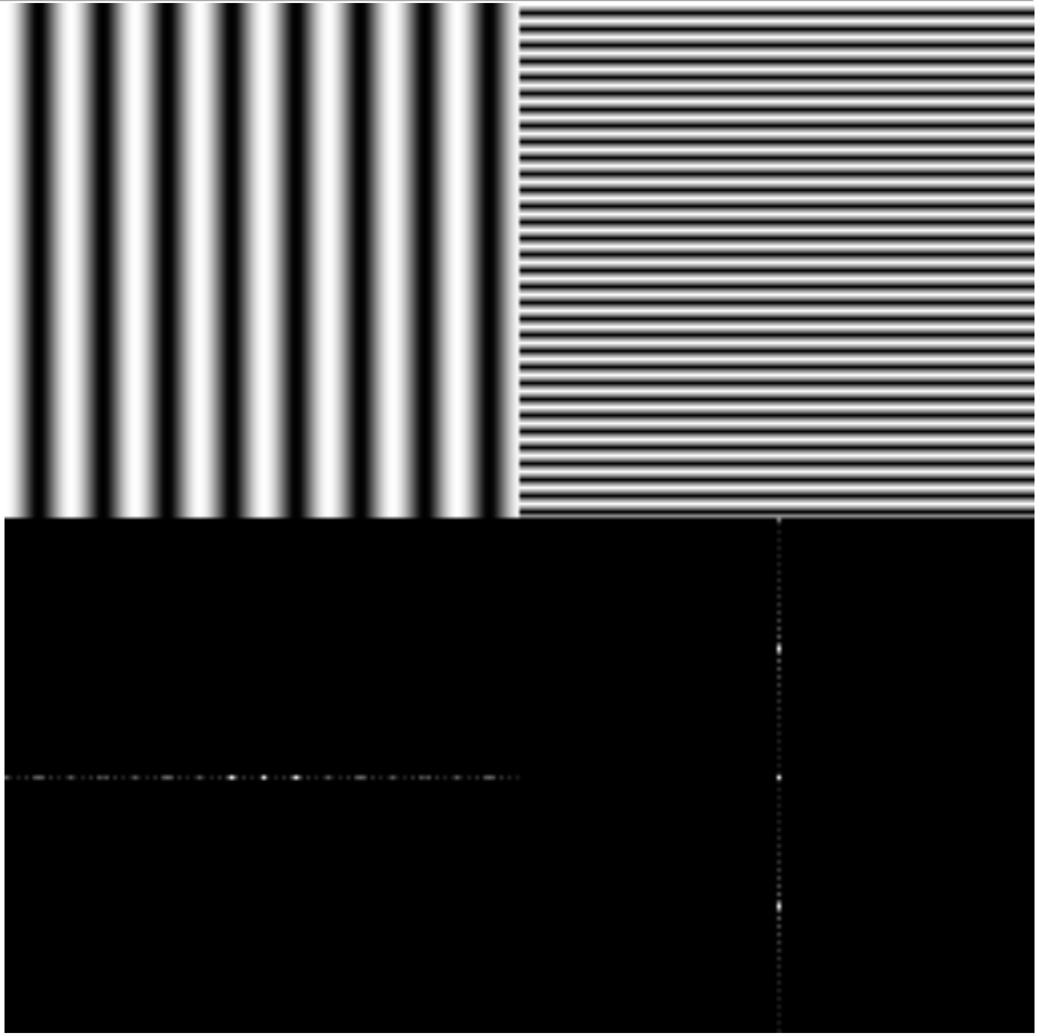
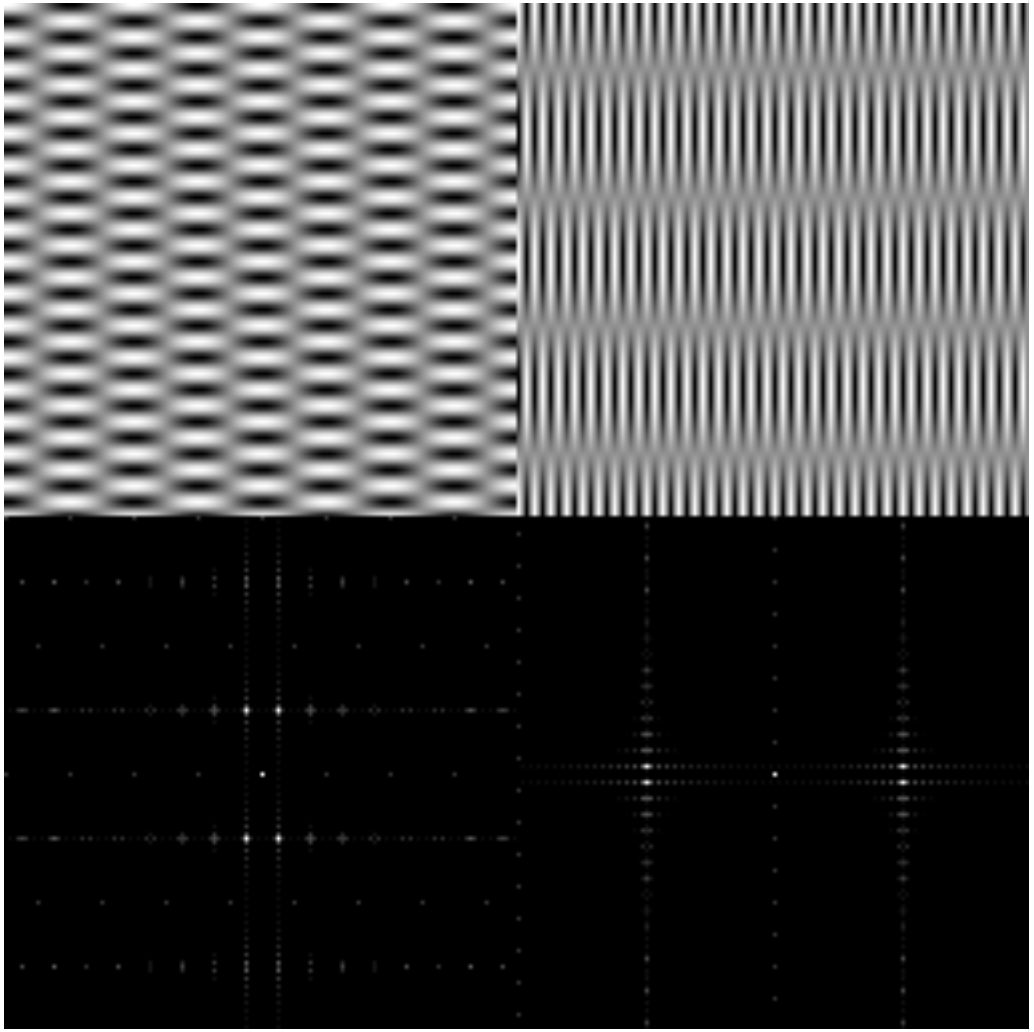


Image Credit: John M Brayer

Pictorial Examples

- Left has 4 cycles horizontally and 16 cycles vertically
- Right has 32 cycles horizontally and 2 cycles vertically
- Note independence of two axes



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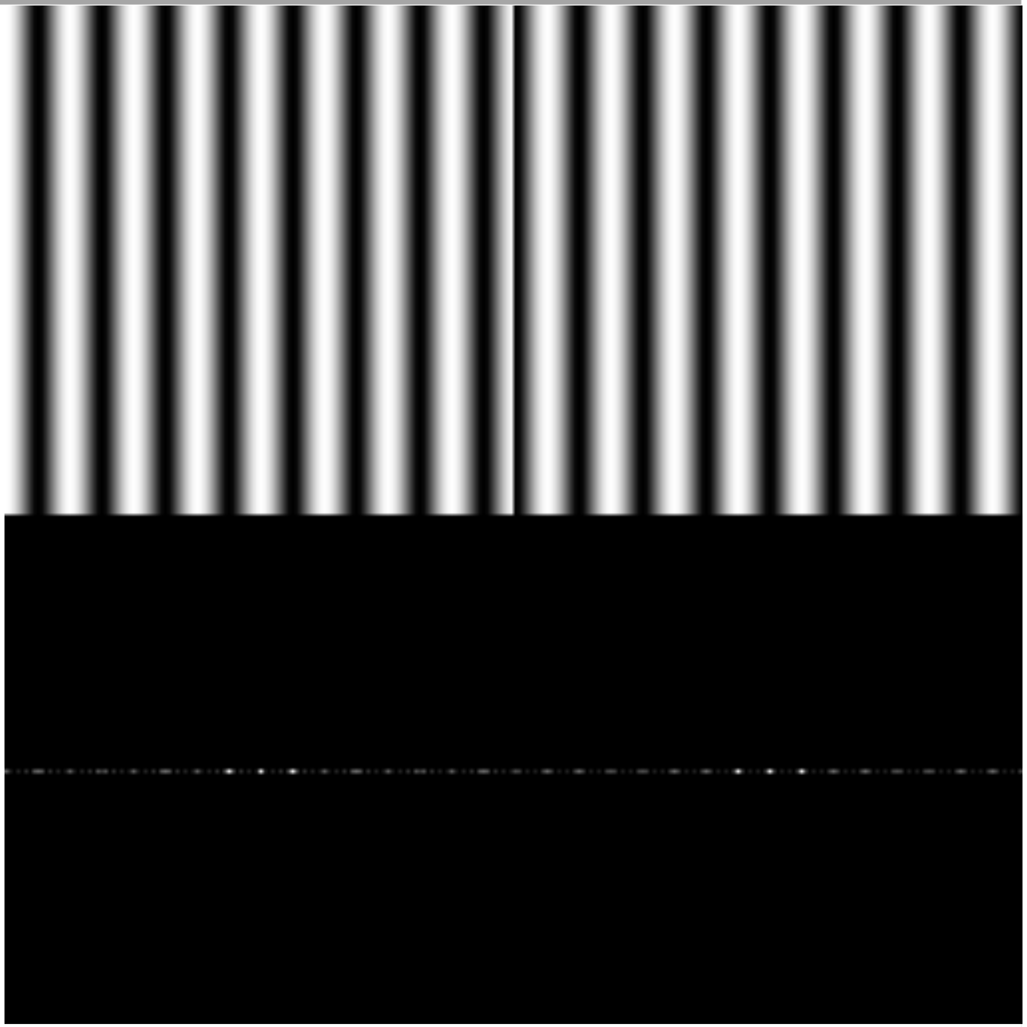
Pictorial Examples

- Left has 8 cycle cosine horizontally
- Right has 8 cycle cosine horizontally shifted by π radians
- **Amplitude** is identical, only measures how much power on different scales
- **Phase** tells you where the structures are
- Phase images only simple for very simple structures (one or few point sources)



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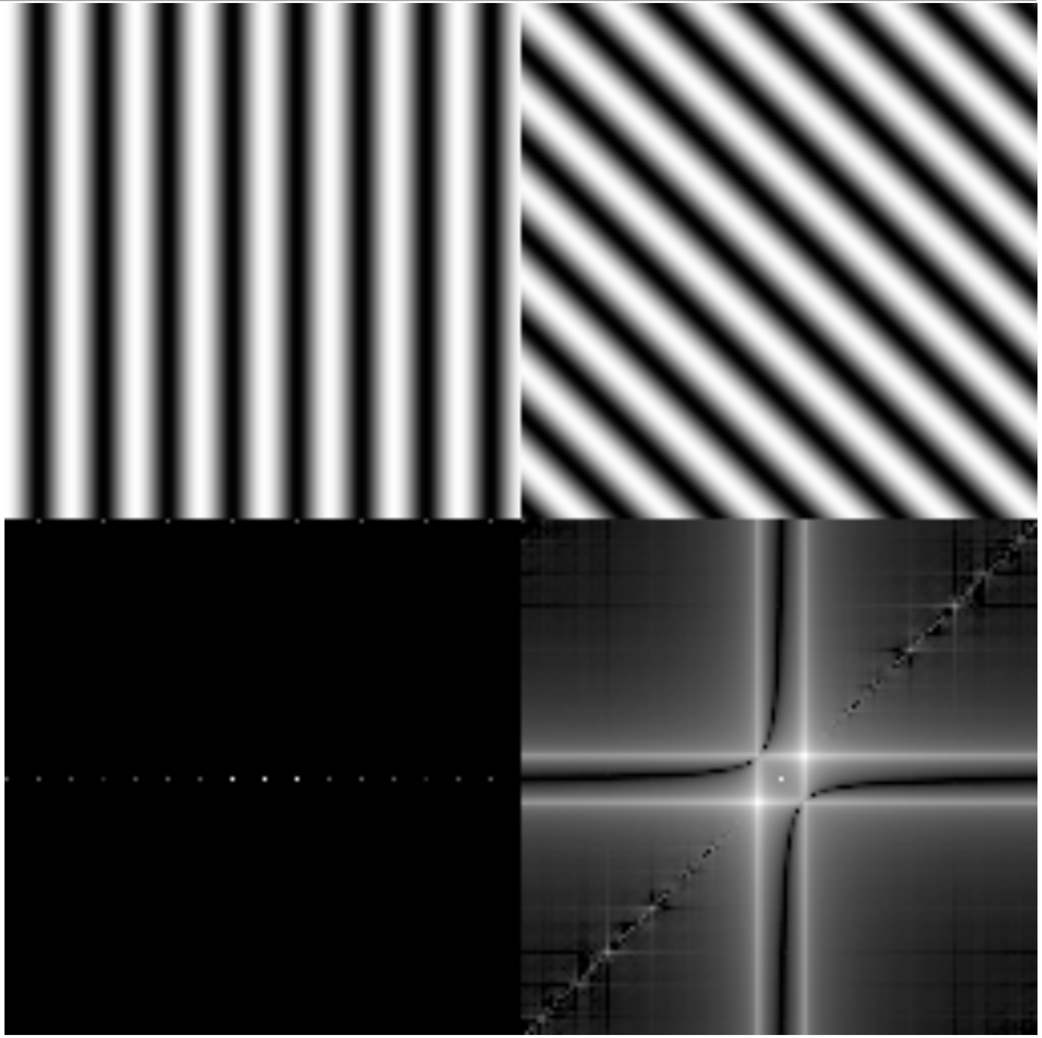
Pictorial Examples

- Left has 8 cycles horizontally
- Right has 8 cycles oriented at 45 degrees
- Why no simple correspondence of the FT's?
- Edge effects!
- Fourier Transform treats image as part of infinite series of identical images in two dimensions



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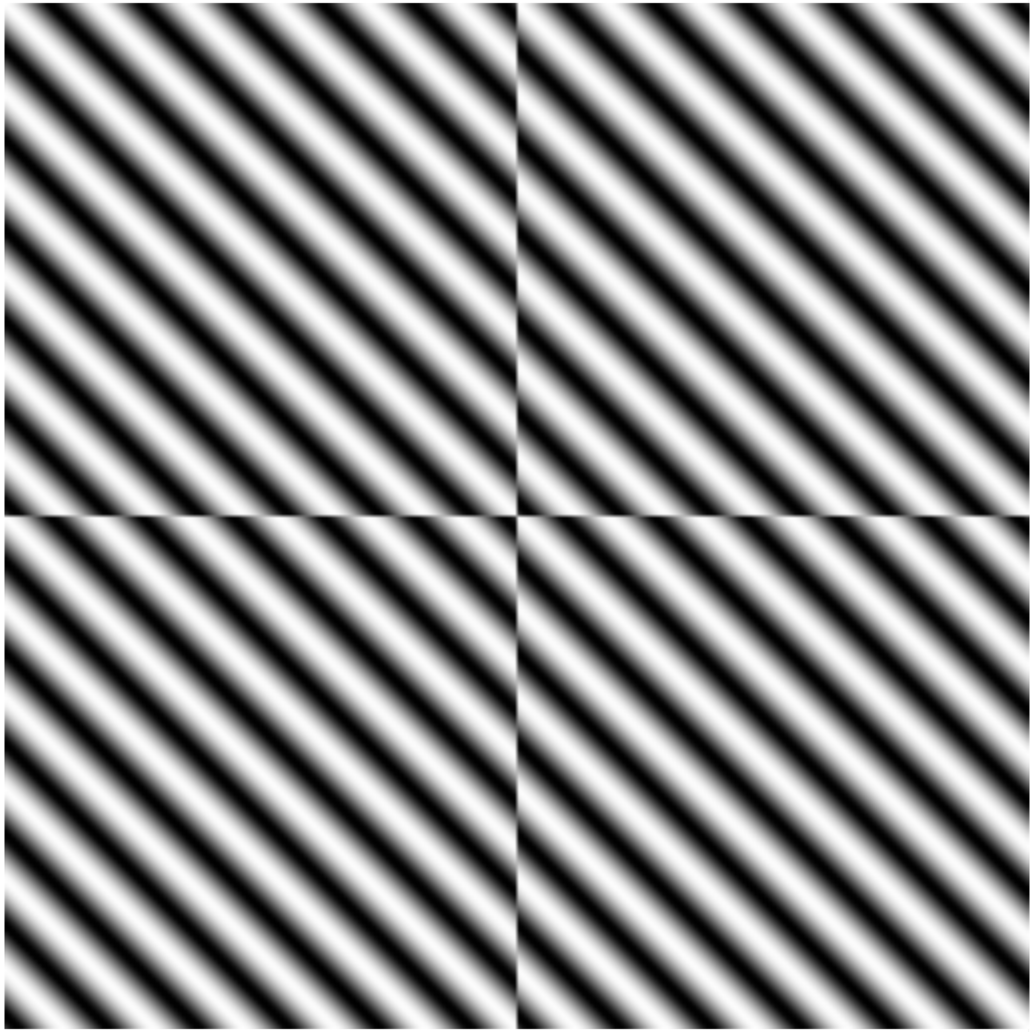

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Pictorial Examples

- Left is tapered image and FT
- Right is un-tapered FT (top) and ideal FT (bottom)
- Tapering can significantly reduce edge effects
- Padding with zeros (by factor of at least two) can eliminate

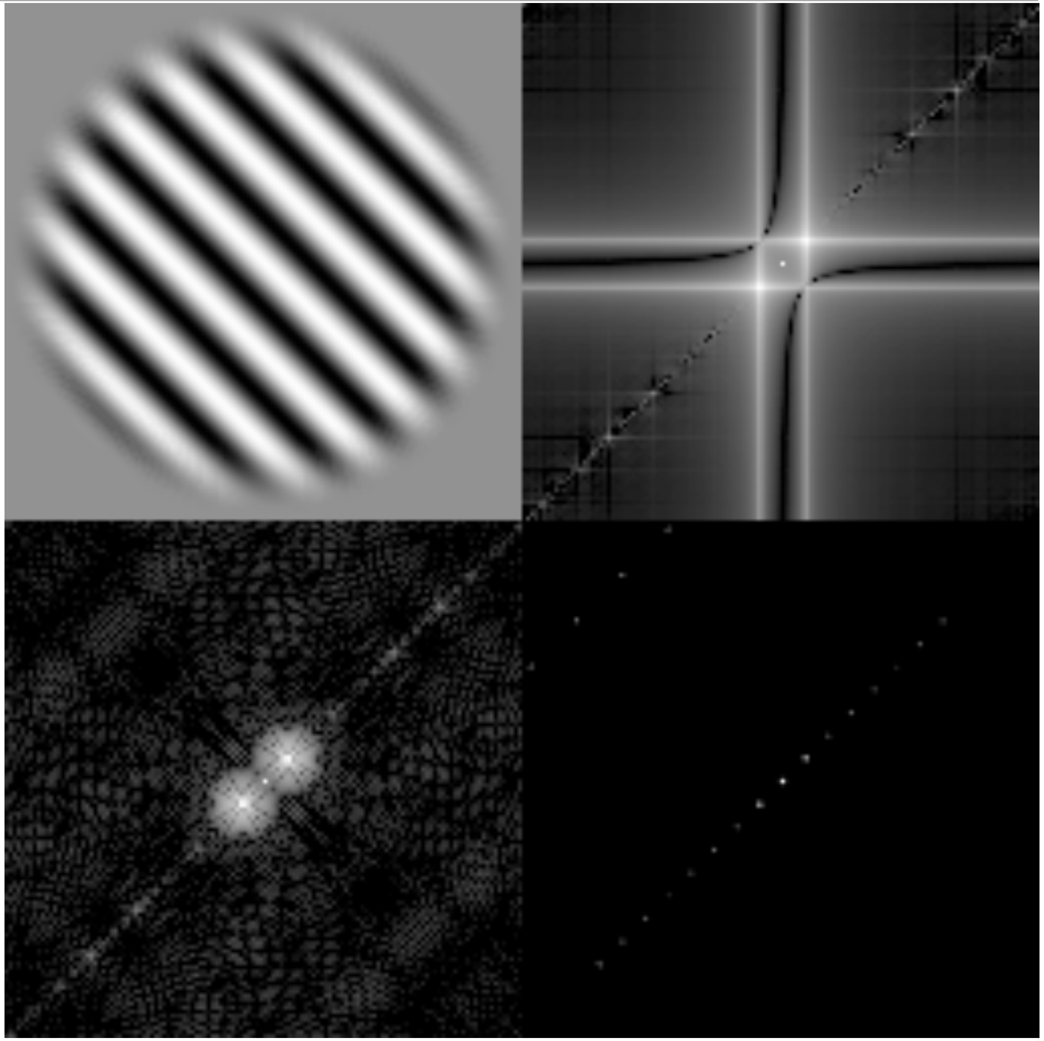
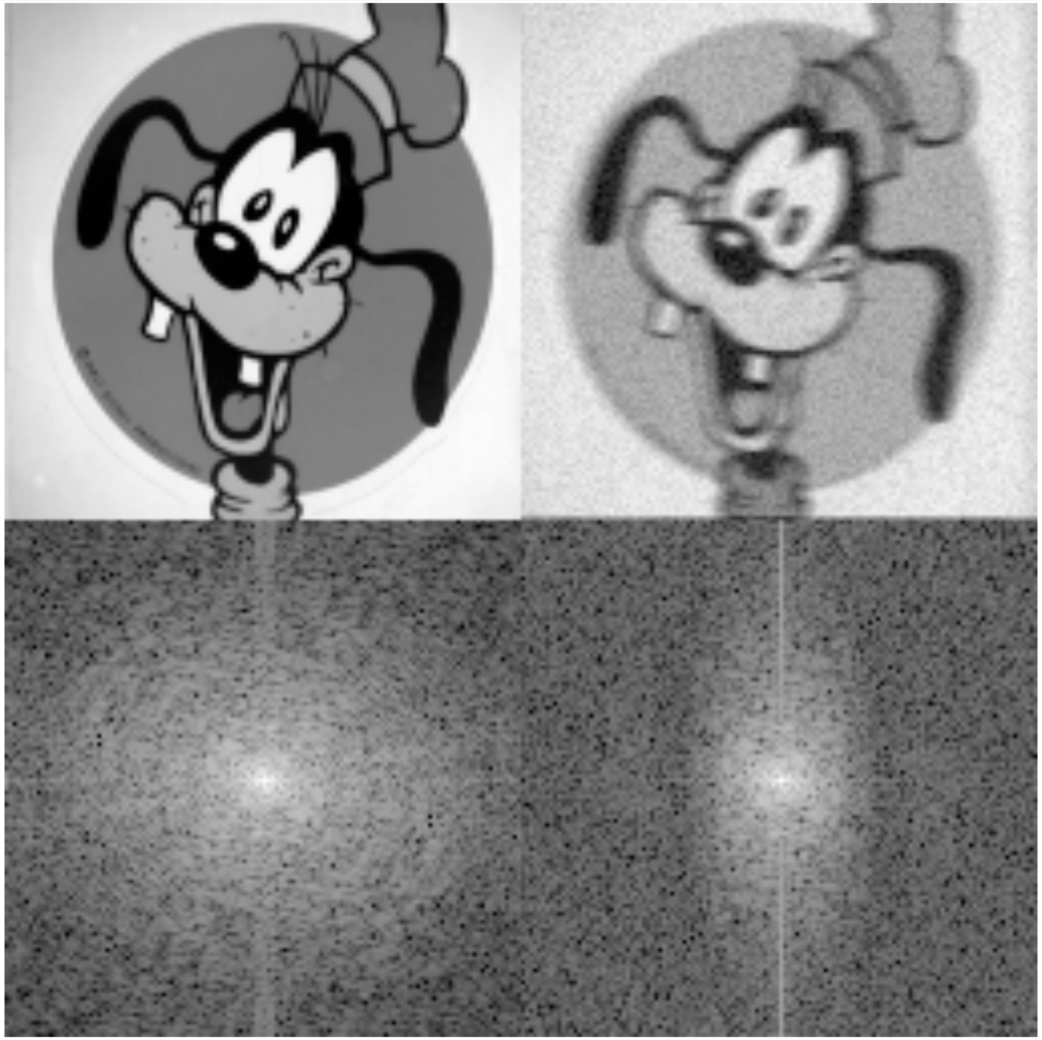


Image Credit: John M Brayer

Pictorial Examples

- Left is full resolution Goofy
- Right is degraded Goofy (horizontal smoothing plus added white noise)
- Vertical and horizontal bands are edge effects from imperfect matching
- Smoothing apparent in FT as diminished extent
- White noise in image gives white noise in FT



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Pictorial Examples

- Periodic image structures in brick wall have FT counterparts
- Horizontal lines have simple counterpart on vertical axis, vertical lines more complex
- Cube faces each have sharp edges with particular orientation
- Each high contrast edge has perpendicular counter -part in FT

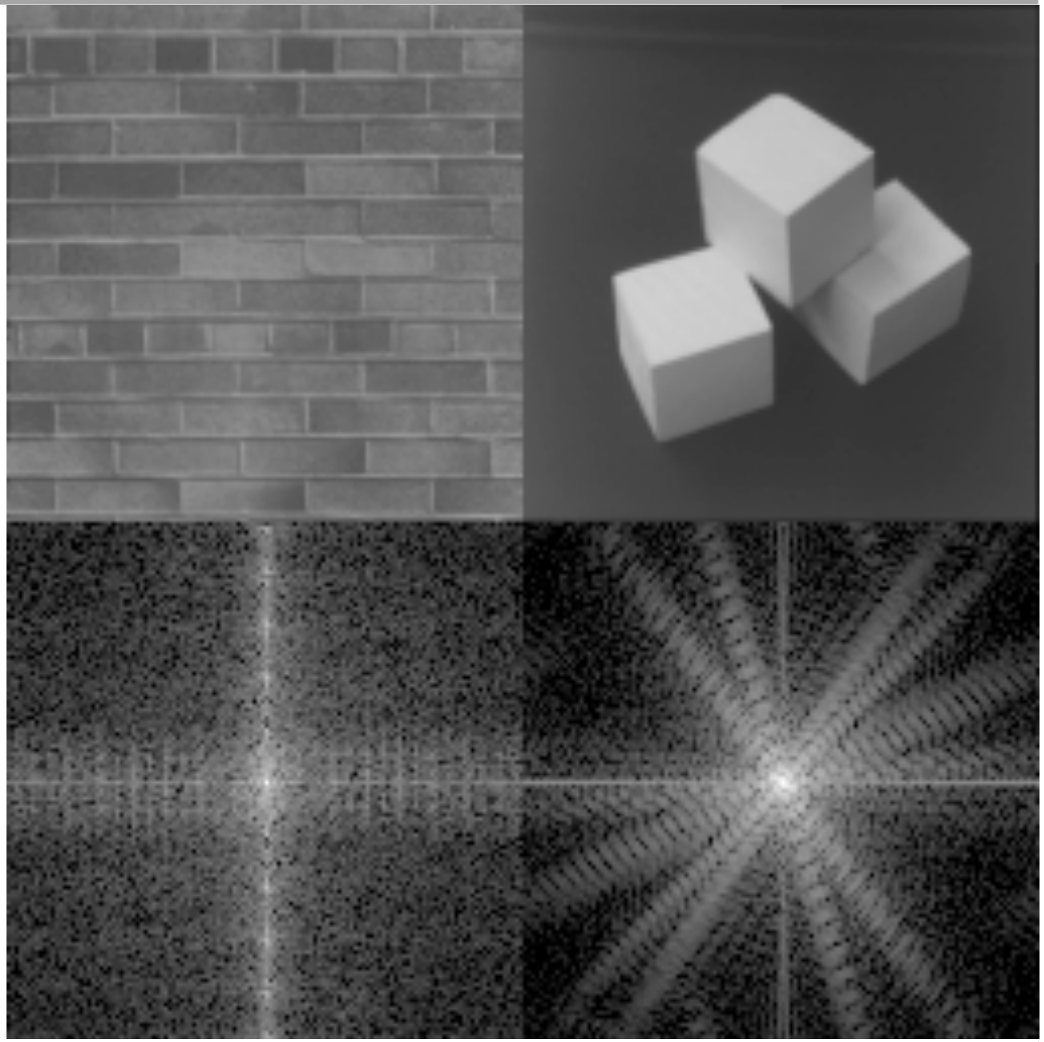
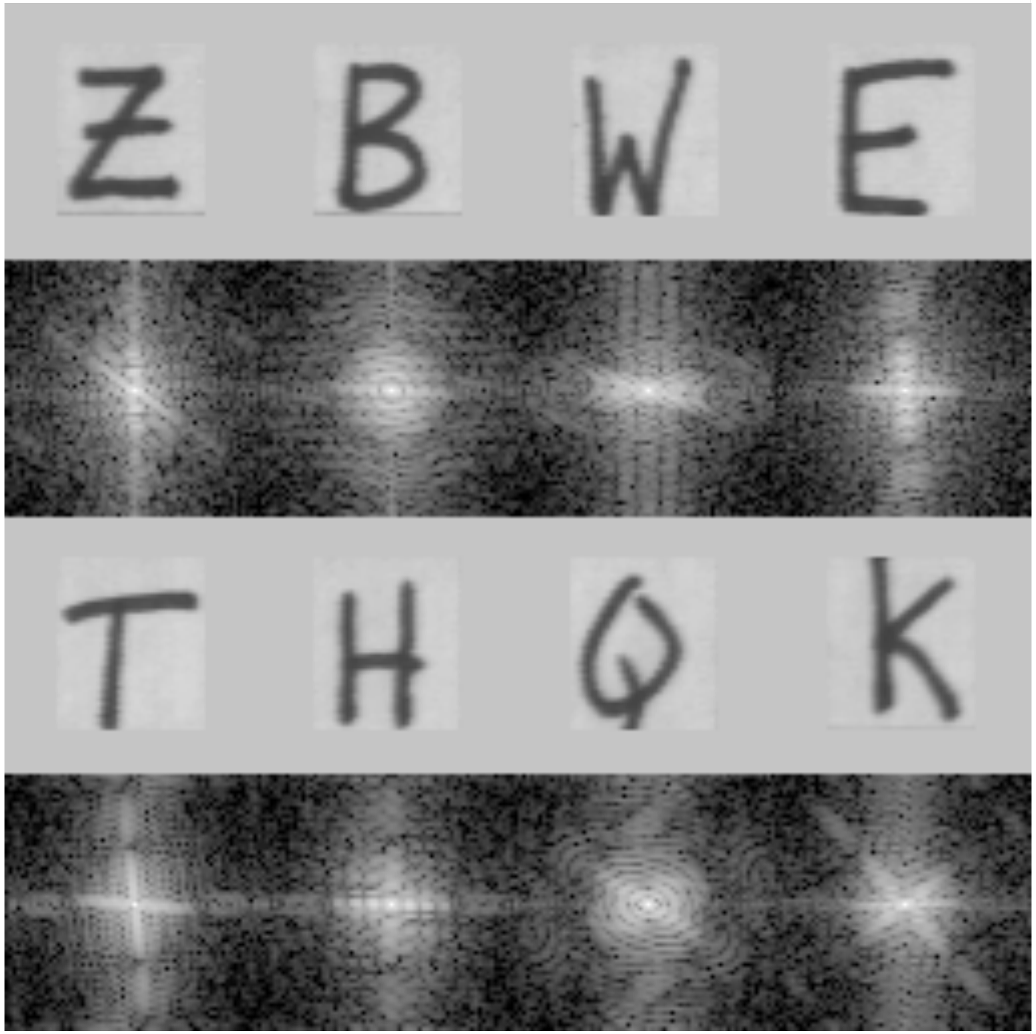


Image Credit: John M Brayer

Pictorial Examples

- Letter/FT pairs demonstrate line segments give perpendicular line segment counterpart
- Circular regions have circular counterparts



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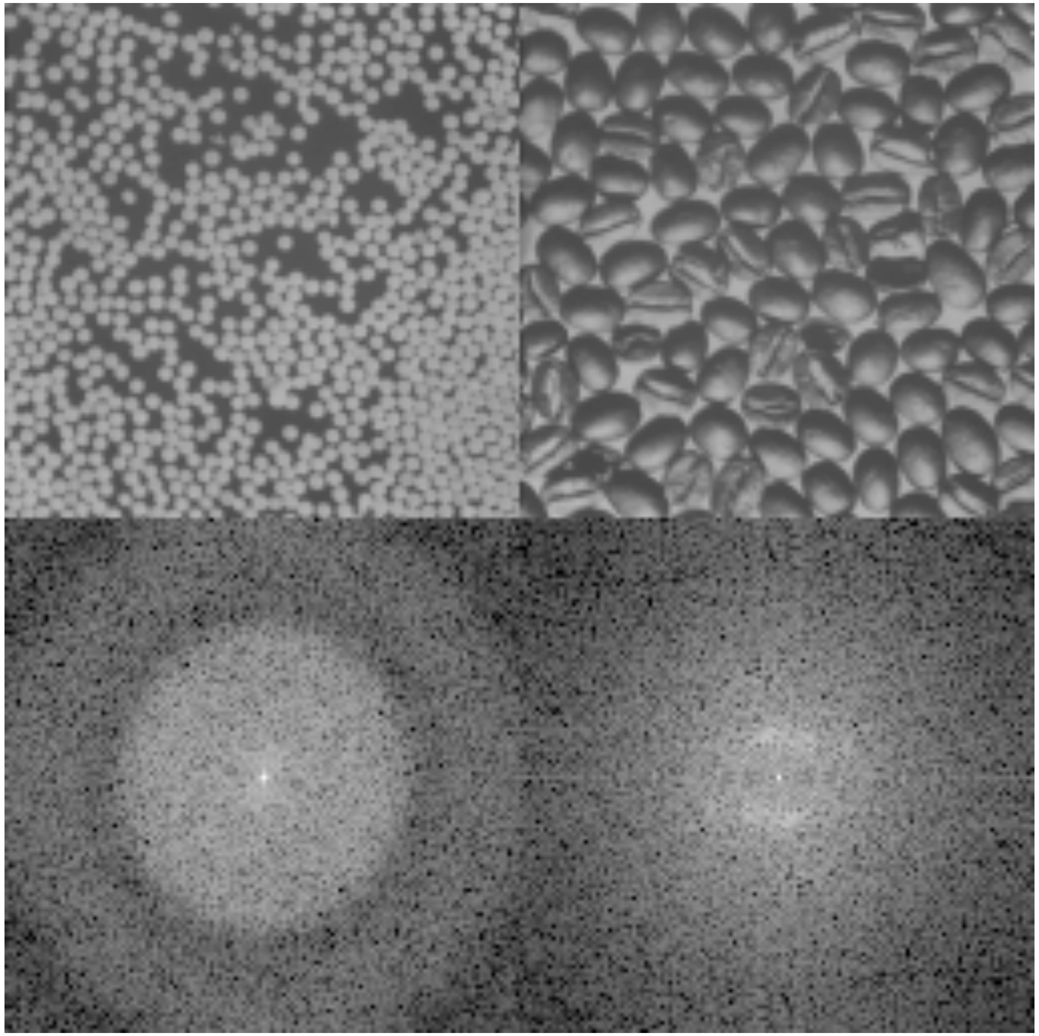
Pictorial Examples

- Collection of identical pellets gives strong signal of (sum of many) individual pellets (amplitude only is shown, position info in phase)
- Collection of coffee beans less distinct due to variation amongst, but faint “mean” coffee bean signal discernible



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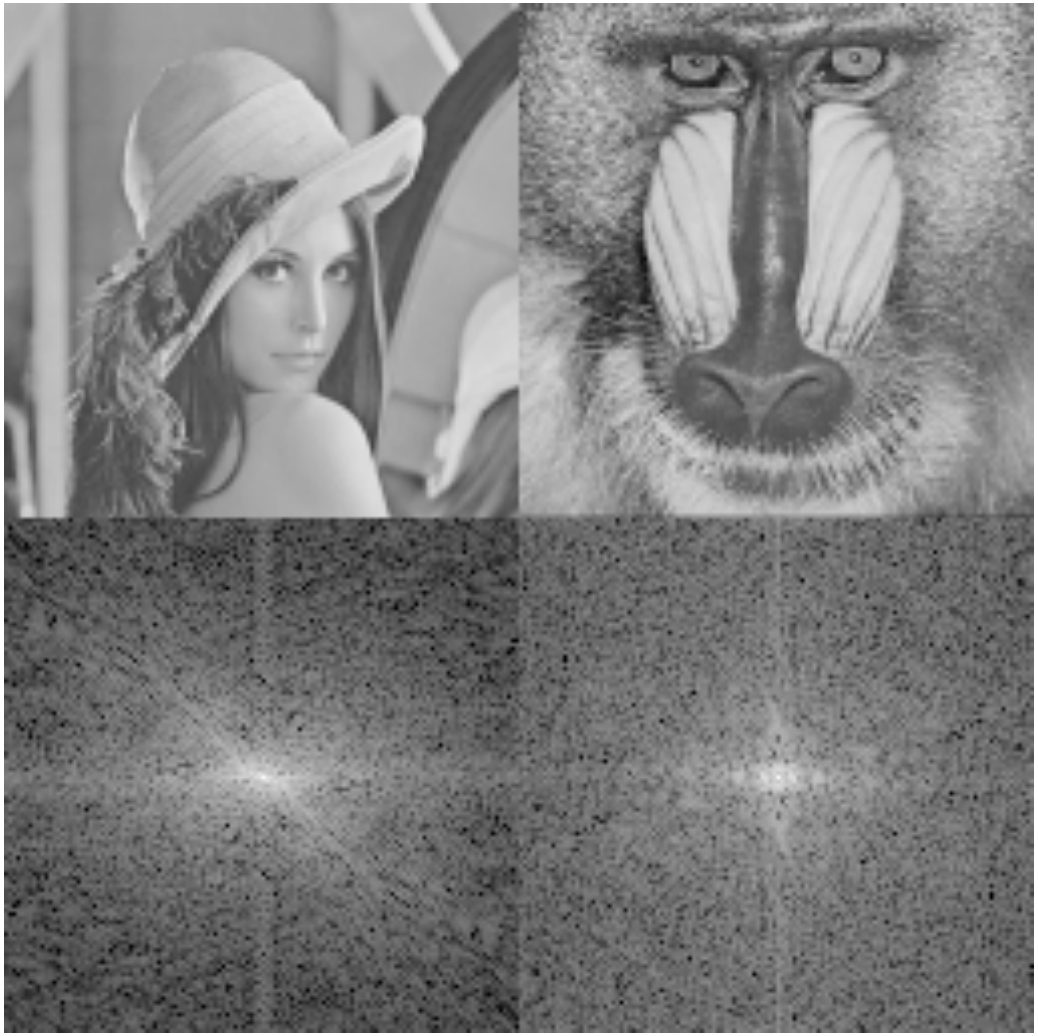
Pictorial Examples

- Complicated objects have complicated FTs
- Edge effects apparent on axes
- Hat brim and background structure give perpendicular diagonal
- Fine hairs give plenty of high spatial frequency power



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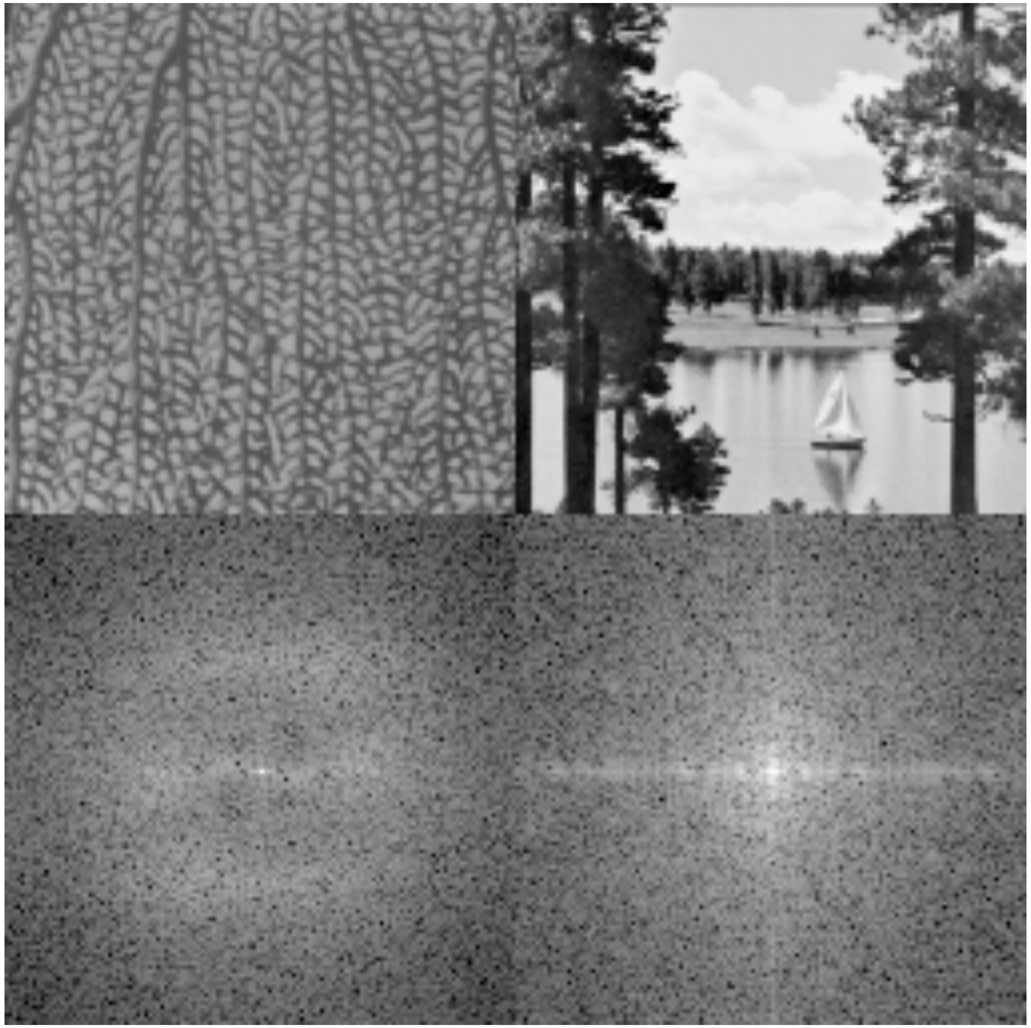
Pictorial Examples

- Seafan image has many similar structures of only a few pixels in extent plus many vertical lines and only few horizontal
- Lake image dominated by vertical tree trunks



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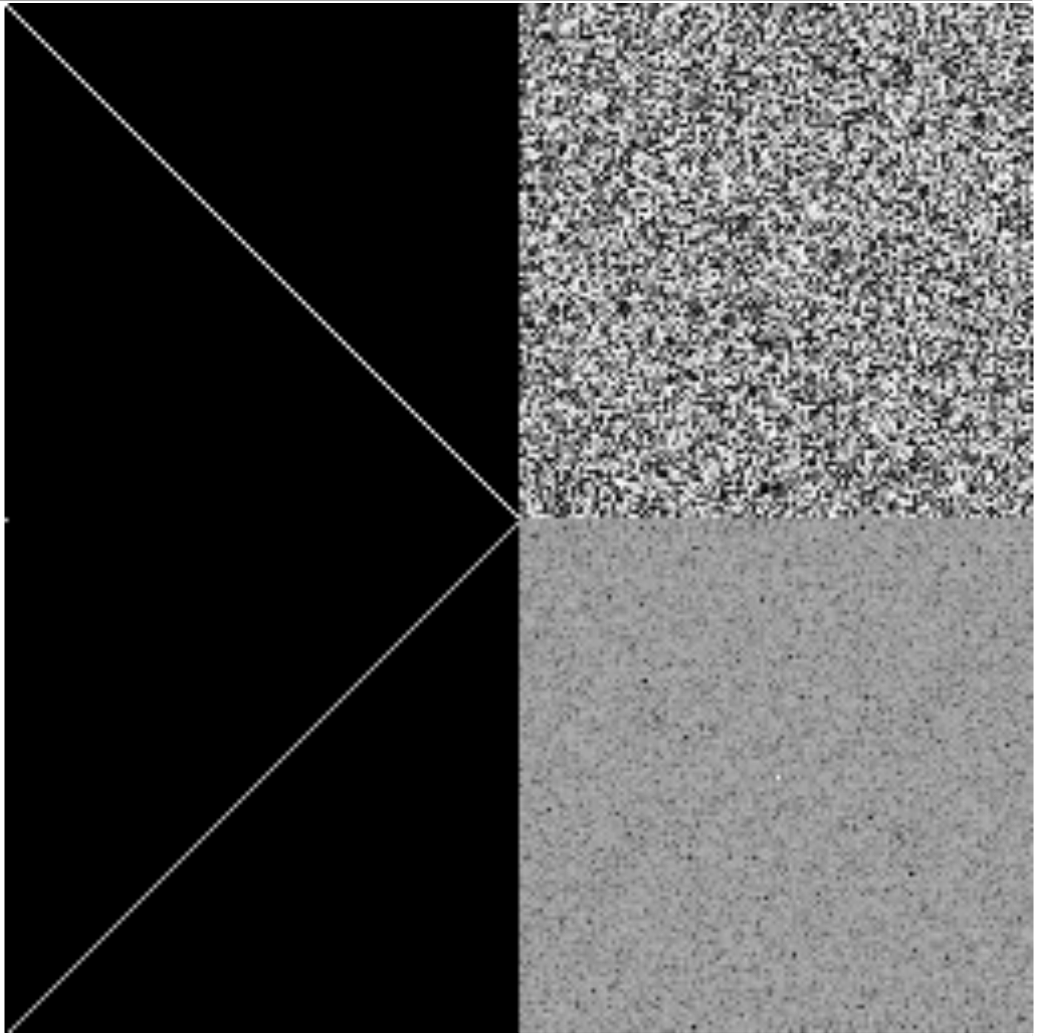
Pictorial Examples

- Extreme case of perpendiculars on left
- Noise gives noise on right



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Pictorial Examples

- Low-pass filtering gives smoothed image plus spatial side-lobes at edges from truncation

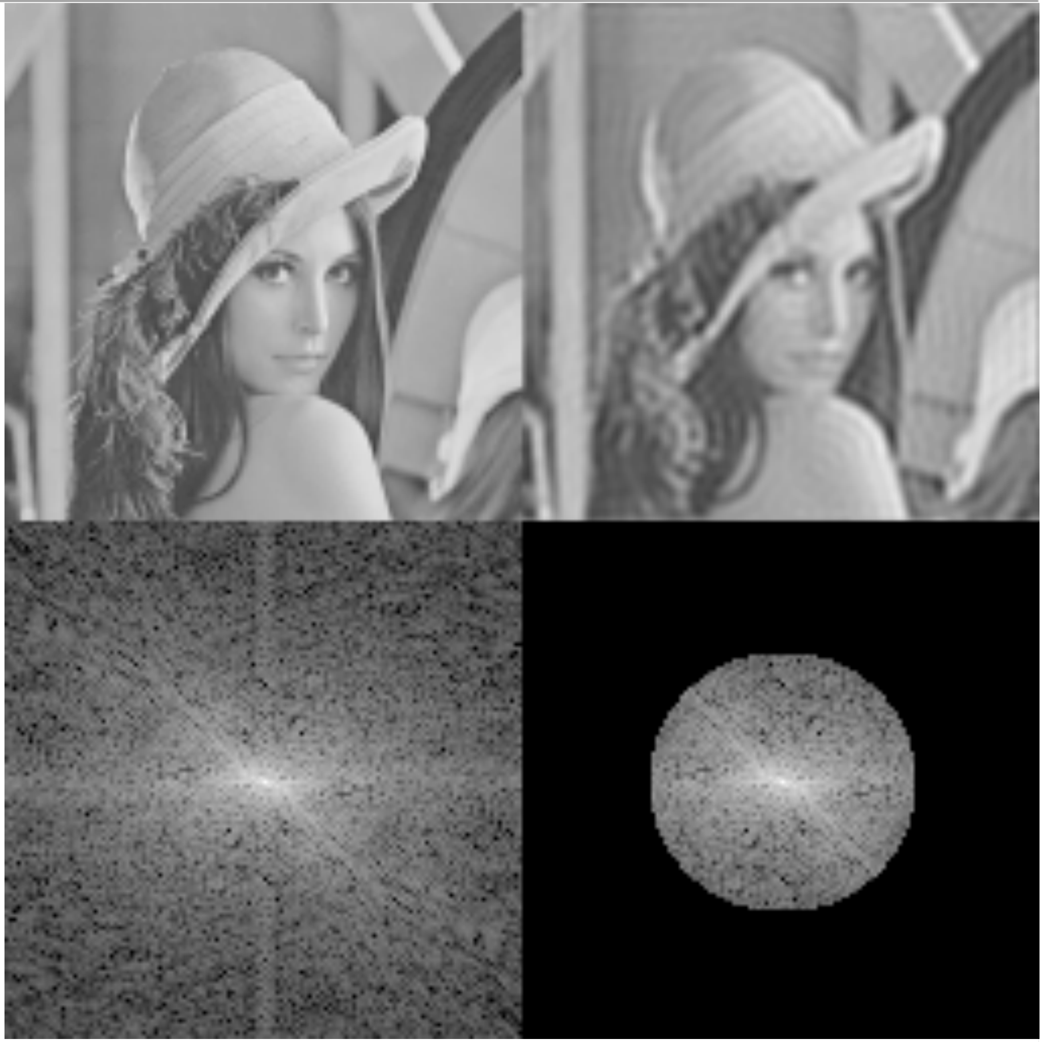

Image Credit: John M Brayer

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Pictorial Examples

- High-pass filter gives "edge detection" but complete loss of total power

Image Credit: John M Brayer

Pictorial Examples

- Radial re-weighting of FT to enhance high spatial freqs relative to low maximizes “sharpness”

Image Credit: John M Brayer

Pictorial Examples

- Goofy image has superposed 4 cycle horizontal cosine
- Point-wise filtering removes the ripple

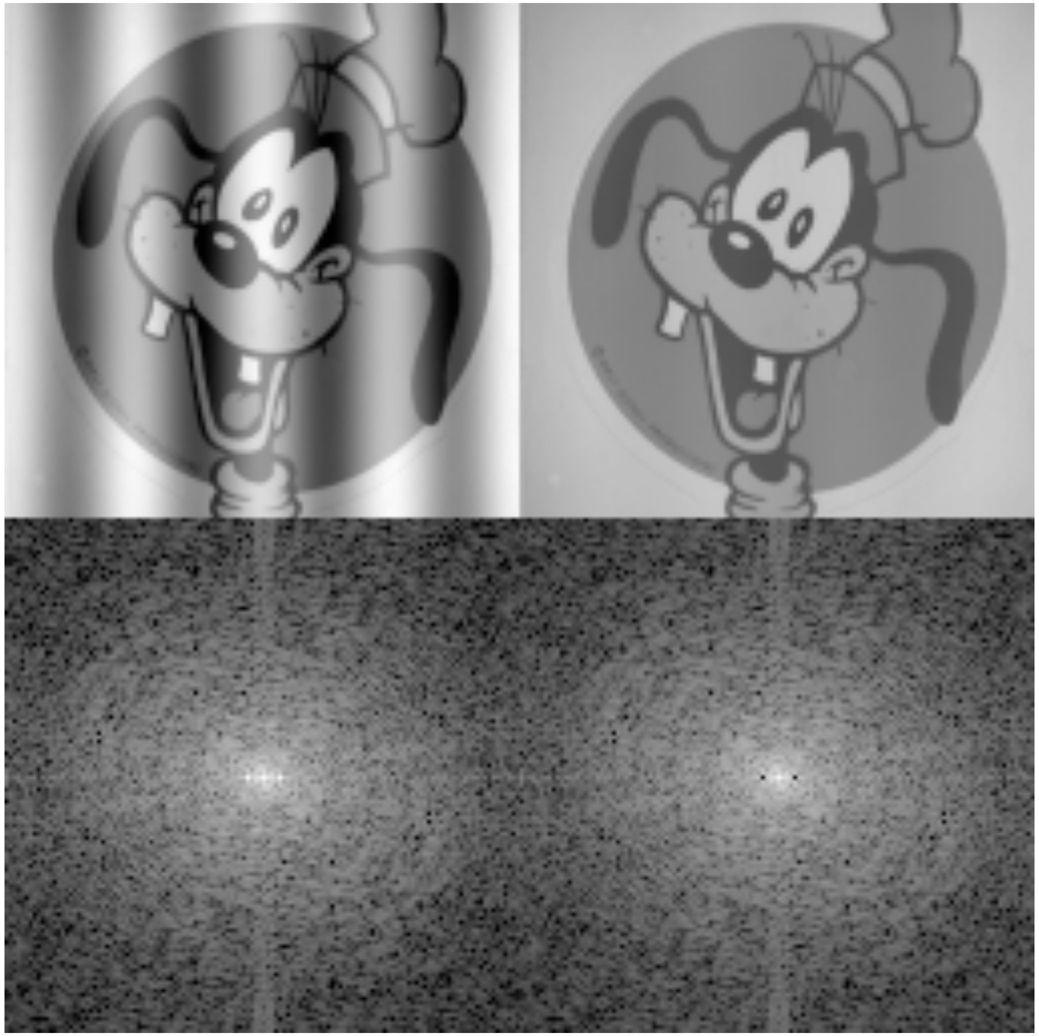

Image Credit: John M Brayer

Pictorial Examples

- Goofy image has grid superposed
- Manual blanking of grid FT signature gives suppression of image -plane grid (only left -side of FT plane is used for FT back to image plane in this FT implementation, ie. assume real image /Hermitian FT)

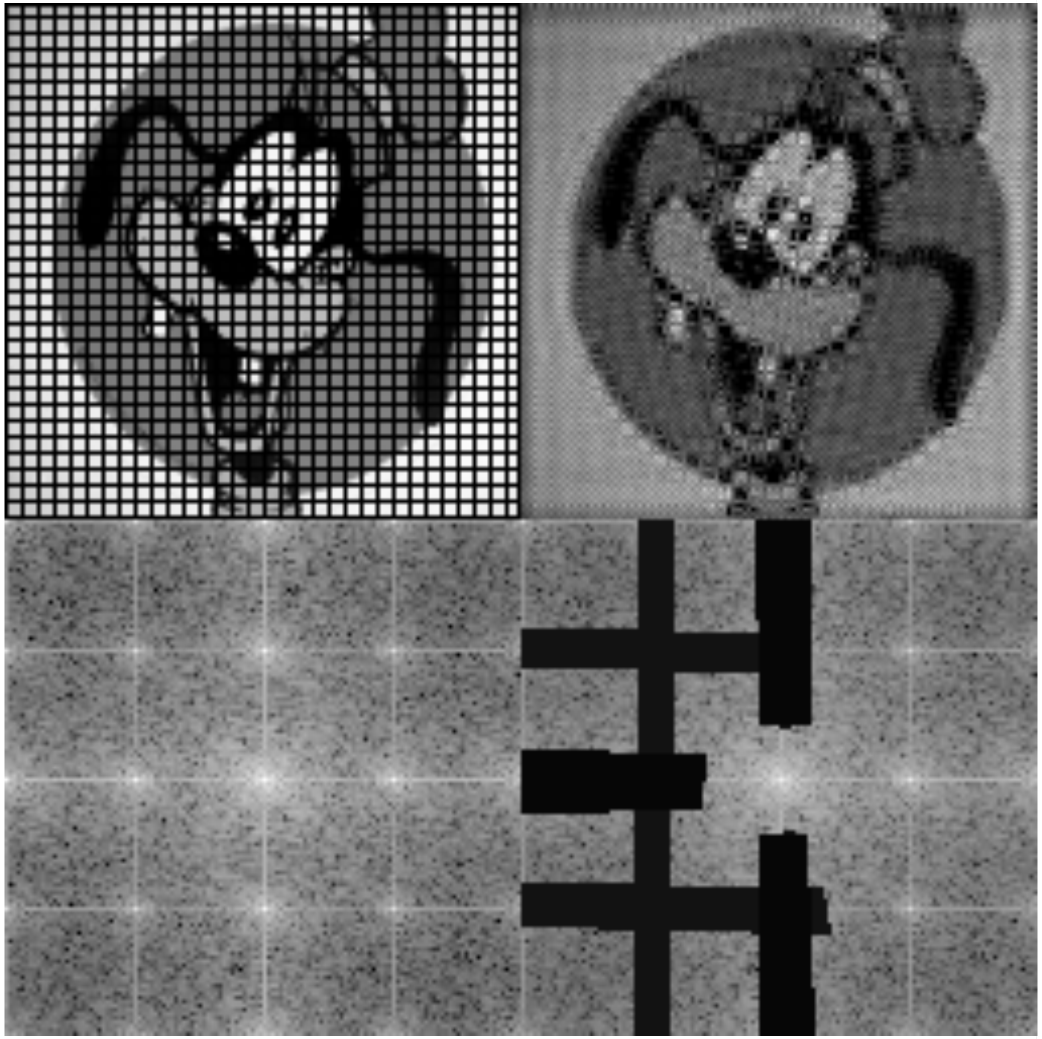

Image Credit: John M Brayer

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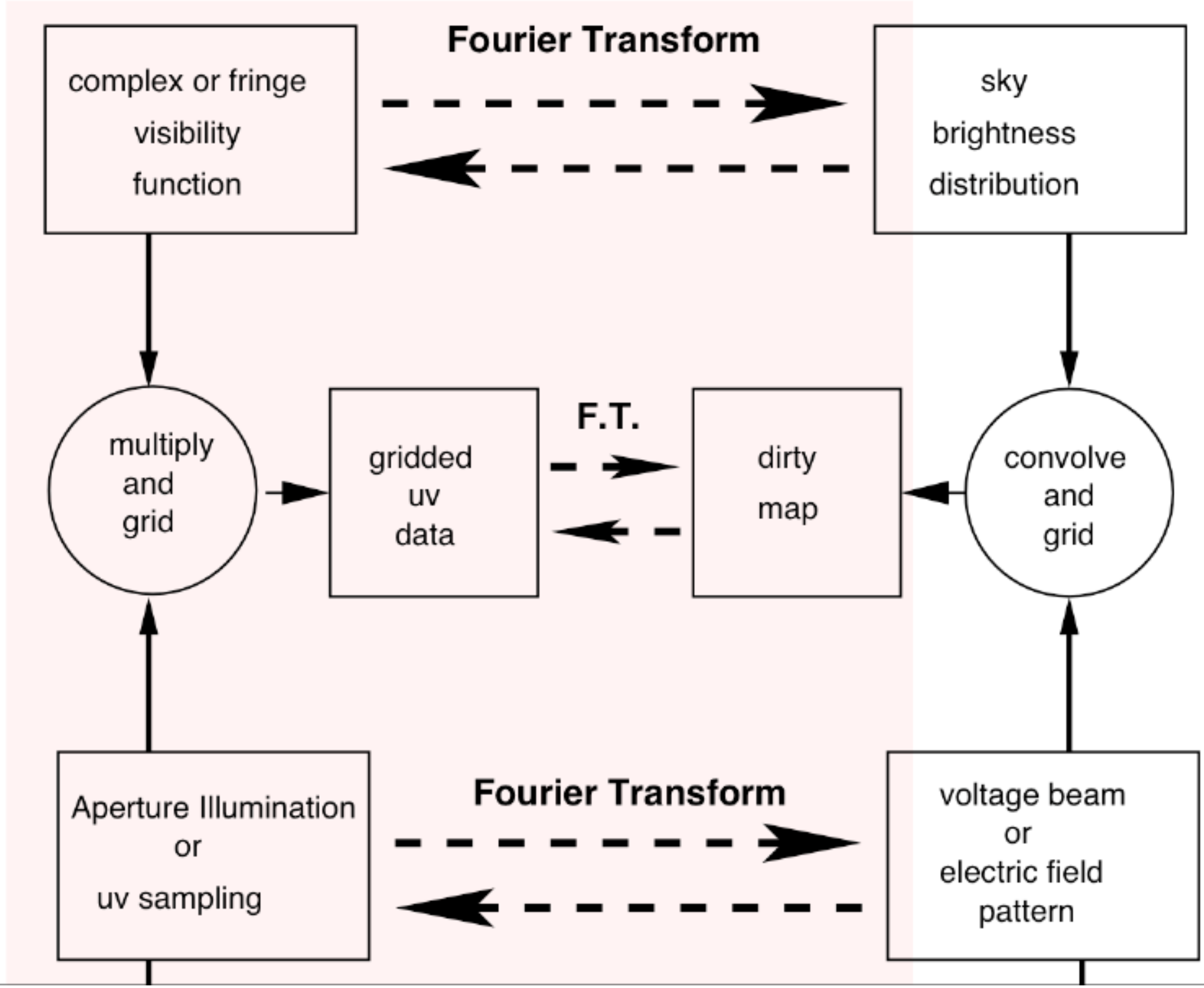
What have we learned?

- **Fourier Transforms are amazingly useful**
- **Basic Properties**
  - Linearity
  - Scale change
  - Delay and Modulation
  - Differentiation and Integration
  - Multiplication and Convolution
  - Parseval's Theorem
- **Pitfalls of discrete and finite sampling**
  - Aliasing
  - Leakage
- **What's next?**
  - Play with them yourself!

How this fits in with the rest of the week:



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