Netherlands Institute for Radio Astronomy

Principles of Interferometry

Part II

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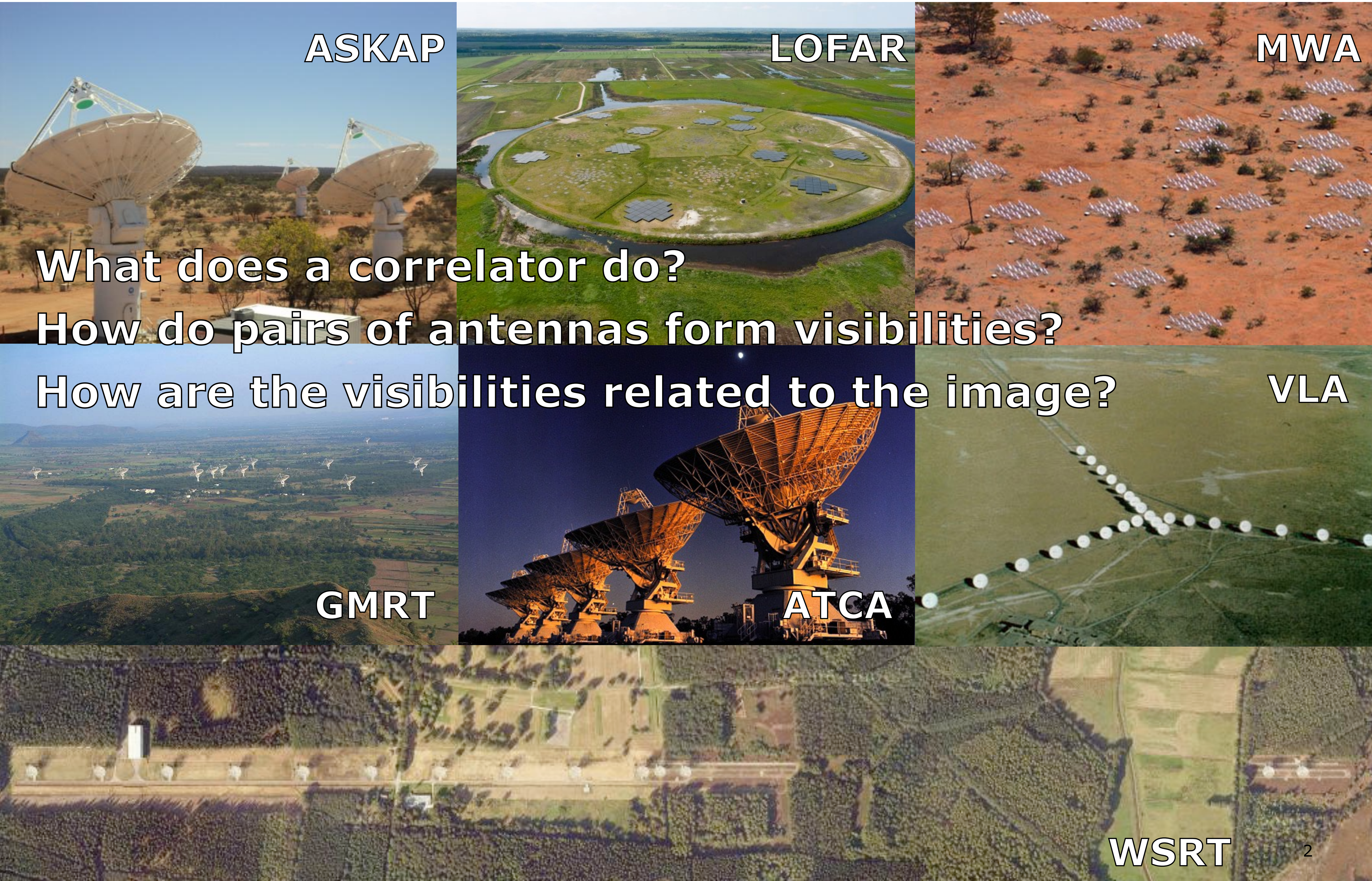
Outline



What does a correlator do?

How do pairs of antennas form visibilities?

How are the visibilities related to the image?

Key concepts

1. Aperture synthesis is used to increase angular resolution with small antennas
2. Correlation takes place by multiplying and time-averaging antenna voltages
3. Each baseline instantaneously measures the visibility function at a single location in the uv plane
4. Earth rotation is exploited to fill the uv plane azimuthally, and bandwidth is exploited to fill the uv plane radially
5. The visibility function is related to the intensity distribution on the sky via a Fourier transform relation
6. The Measurement Equation is a useful tool for understanding the instrumental connection between visibilities and the sky brightness

Far-field EM radiation

- Definition of wavefront: a surface of constant phase - perpendicular to k

From Griffiths "Intro to EM"

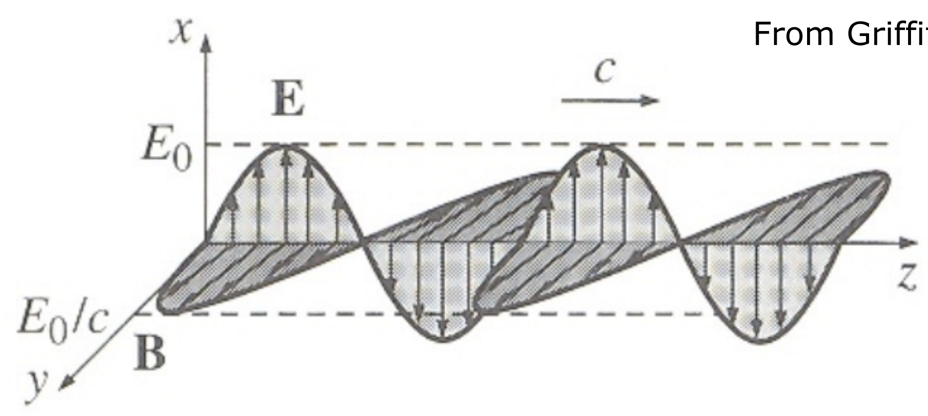


Figure 9.10

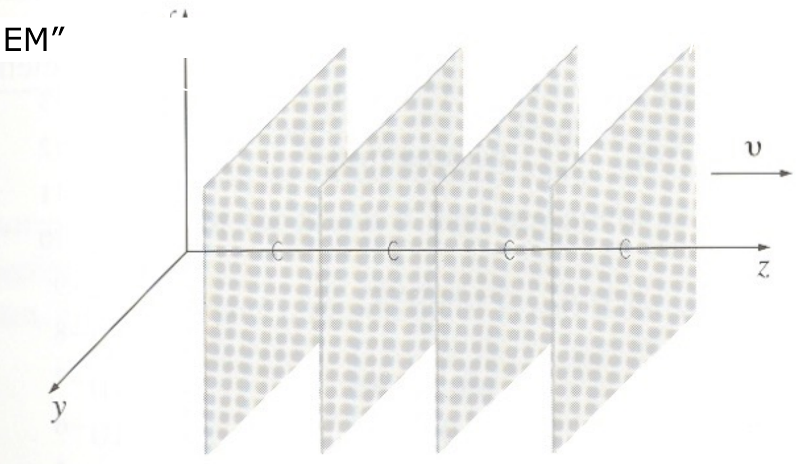


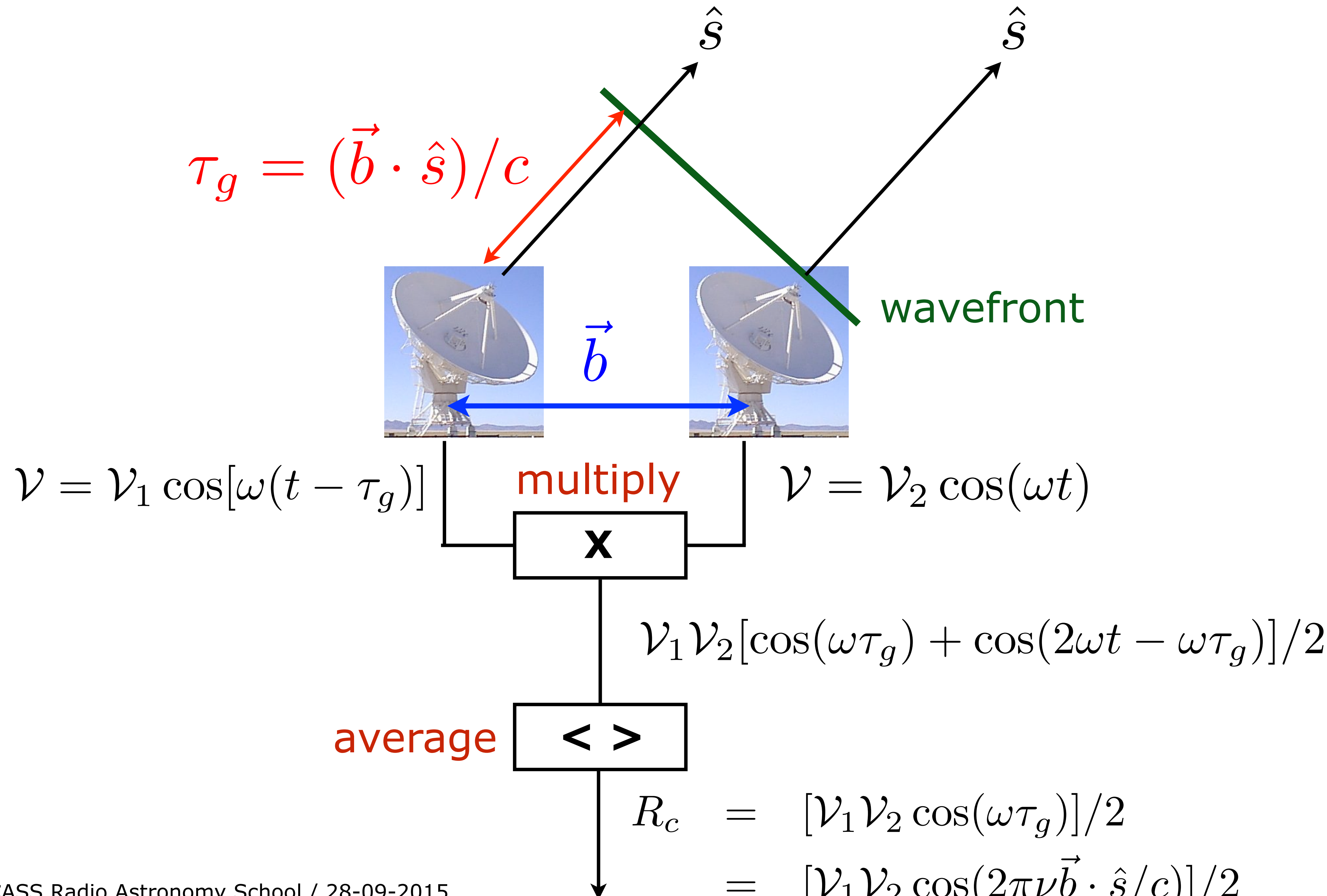
Figure 9.9

Fig.2: Flattening of spherical wavefront to planar wavefront with distance.

- This is the wavefront for a point source (expanding spherical wavefront) in the far field - other (extended) radiation sources have more complicated wavefront structures.

Stationary, Monochromatic Interferometer

- Assume small frequency width ($\Delta\nu$) and no motion of the source.
  Now consider radiation from a small solid angle ($d\Omega$) from direction S



Signal Multiplication

- The two input signals are shown in red and blue
  Their product is shown in black
  The output response (average of black curve) is shown as green dashed line

In phase

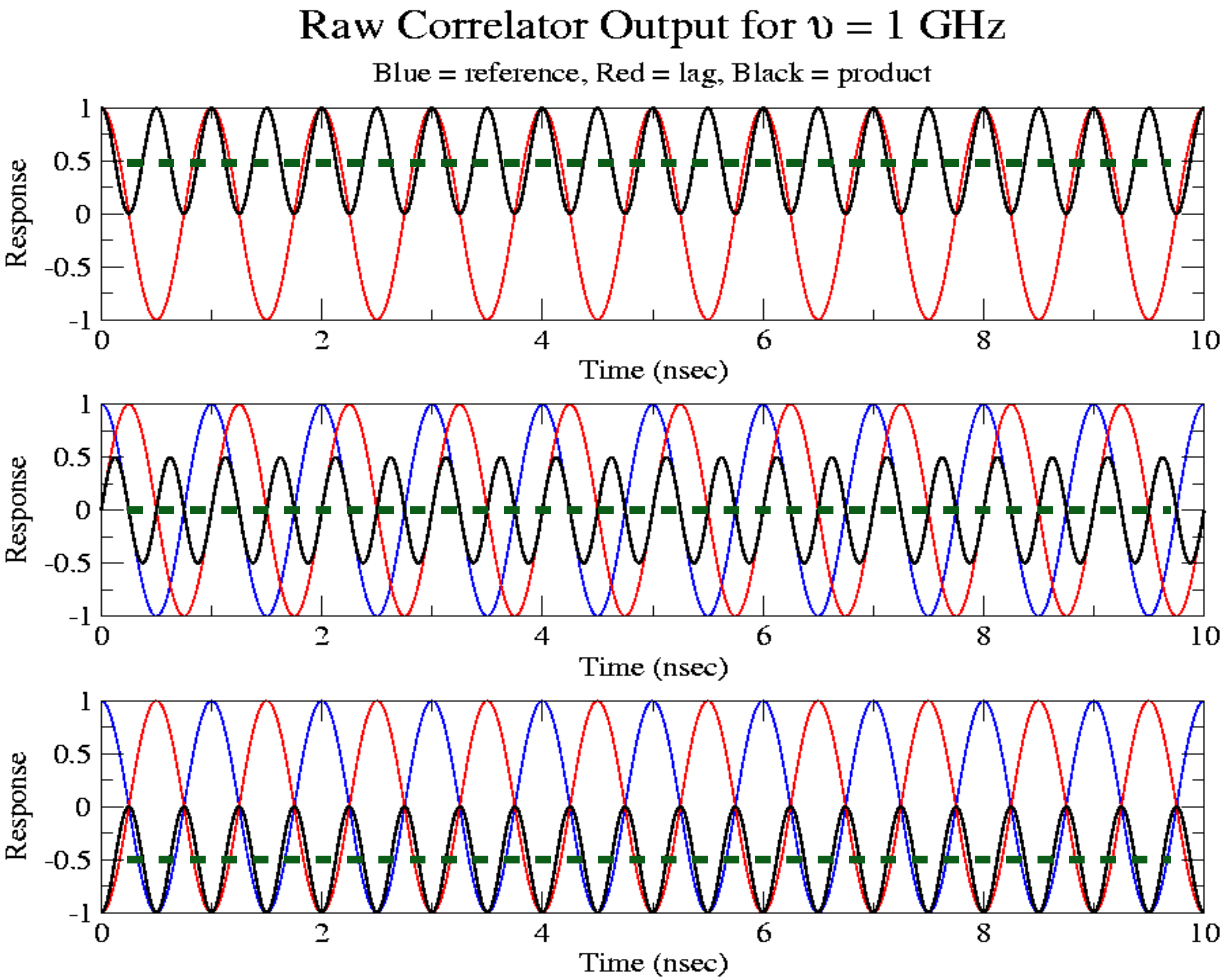
$$\tau_g = n\lambda/c$$

Quadrature phase

$$\tau_g = (2n+1)\lambda/4c$$

Anti-phase

$$\tau_g = (2n+1)\lambda/2c$$



Signal Multiplication

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- The averaged signal is independent of time, but it is dependent on the delay
  - Since the delay is related to the direction, it is related to the source distribution on the sky
  - The source distribution is directly related to the correlator response
- Here we have considered the voltage of the signal.
  How is that related to the source intensity?

$$\mathcal{V} \propto E \propto \sqrt{I}$$

  This means that the product $\mathcal{V}_1 \mathcal{V}_2$ is proportional to the source intensity I which is measured in a unit called the JANSKY

$$10^{-26}\,\mathrm{W\,m^{-2}\,Hz^{-1}\,sr^{-1}} = 1\,\mathrm{Jansky(Jy)}$$

- The strength of the product also depends on the aperture areas of the antennas and the electronic gain factors, but these can be calibrated for (addressed in lecture tomorrow)

- To determine the dependence of the response over an extended object, we integrate over solid angle.

The "cosine" correlator response

- The response from an extended source is just the integral of the response over the solid angle of the sky

$$R_c = \int\int I_\nu(\hat{s}) \cos(2\pi\nu\vec{b}\cdot\hat{s}/c)\, d\Omega$$

(neglecting any frequency dependence)

- NOTE: the vector s is a function of direction, so the phase in the cosine is dependent on the angle of arrival of the wavefront, and thus on the source structure.

Amplitude ⇔ Intensity

Phase ⇔ Position

- Now we have a relationship between the quantity of interest (*I(s)*, the source brightness distribution on the sky) and an observable quantity (R_c, the interferometer response)

Schematic

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- The cosine correlator can be thought of as casting a sinusoidal fringe pattern on the sky (of angular scale λ/b). The correlator multiplies the source intensity distribution by this fringe, and integrates the product over the sky.

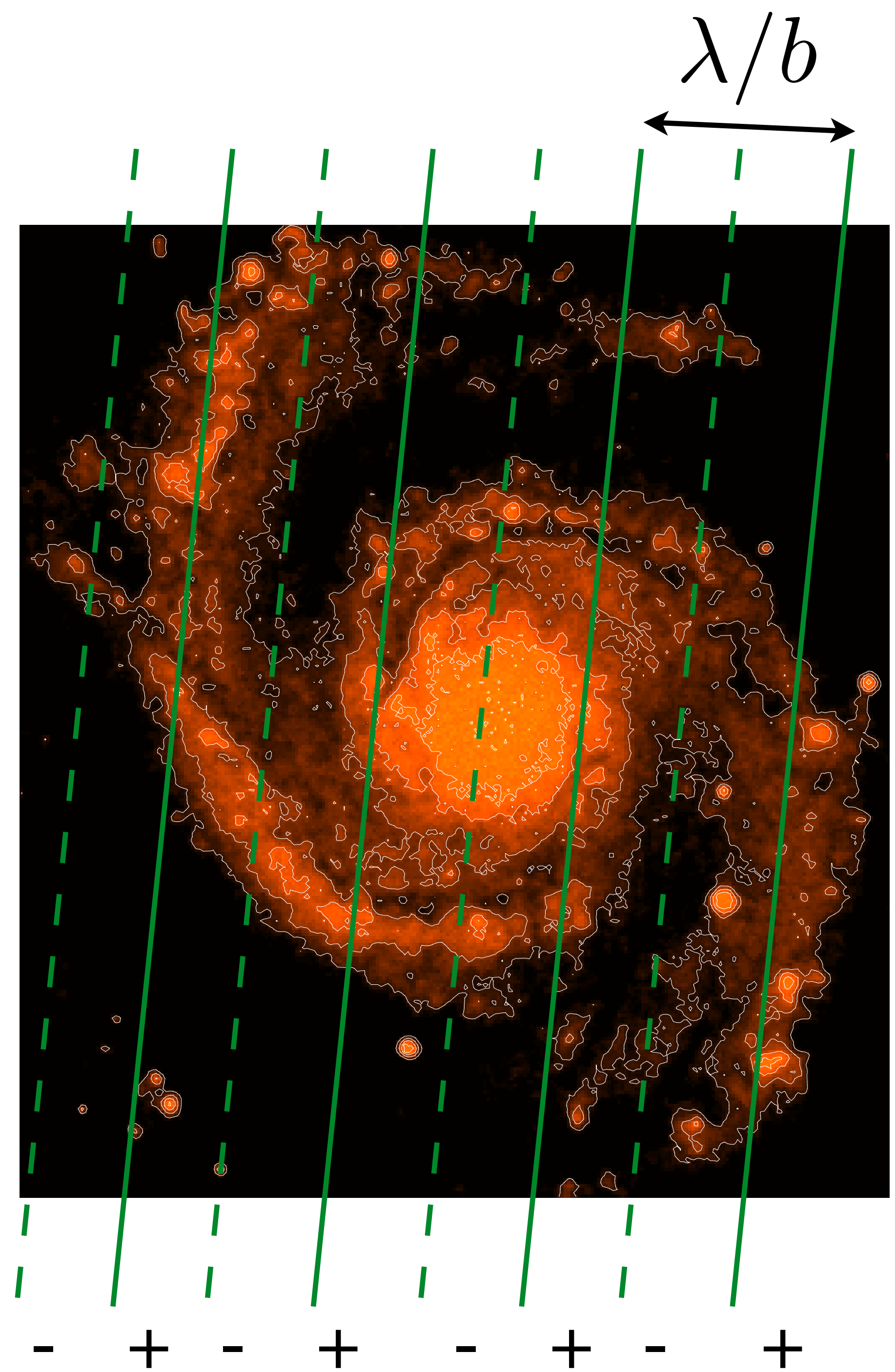
$$R_c = \int\int I_\nu(\hat{s}) \cos(2\pi\nu\vec{b}\cdot\hat{s}/c)\, d\Omega$$

source brightness

fringe pattern

- The orientation of the fringe is set by the baseline geometry

  The fringe separation is set by baseline length, and the observing wavelength



Odd and Even functions

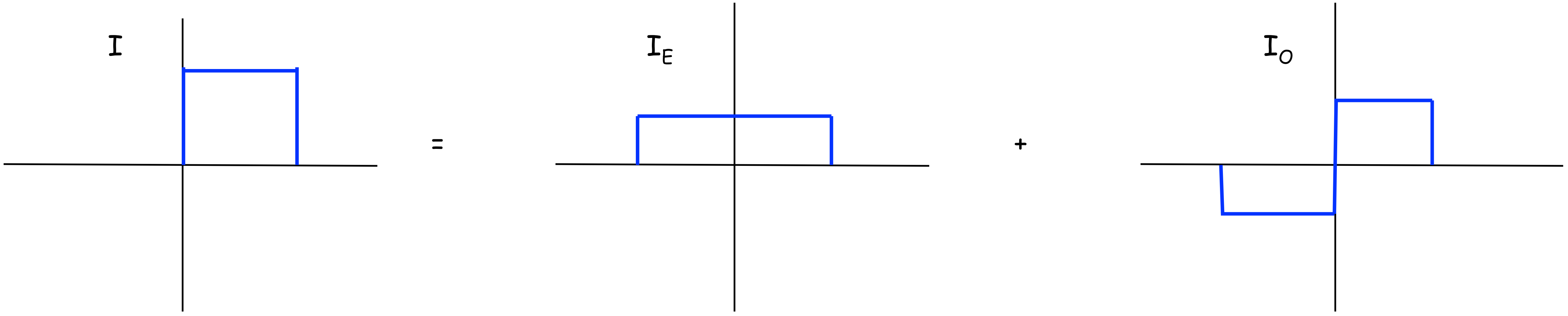
- However, the measured quantity, R_c, is insufficient - it is only sensitive to the even part of the brightness distribution, $I_E(s)$

- Any real function, I, can be expressed as the sum of two real functions with specific symmetries,
  - A part with even symmetry

$$I_E(x,y) = [I(x,y) + I(-x,-y)]/2 = I_E(-x,-y)$$

  - A part with odd symmetry

$$I_O(x,y) = [I(x,y) - I(-x,-y)]/2 = -I_O(-x,-y)$$



Recovering the "odd" part: sin correlator

- The integration of the product of the cosine correlator response over the source brightness distribution is only sensitive to the even part of the distribution

$$R_c = \int\int I_\nu(\hat{s})\cos(2\pi\nu\vec{b}\cdot\hat{s}/c)\,d\Omega = \int\int I_E(\hat{s})\cos(2\pi\nu\vec{b}\cdot\hat{s}/c)\,d\Omega$$

since the integral of the product of an even and an odd function is zero.

- To recover the odd part of the source brightness distribution, we need an odd coherence pattern - we replace the cos with a sin :

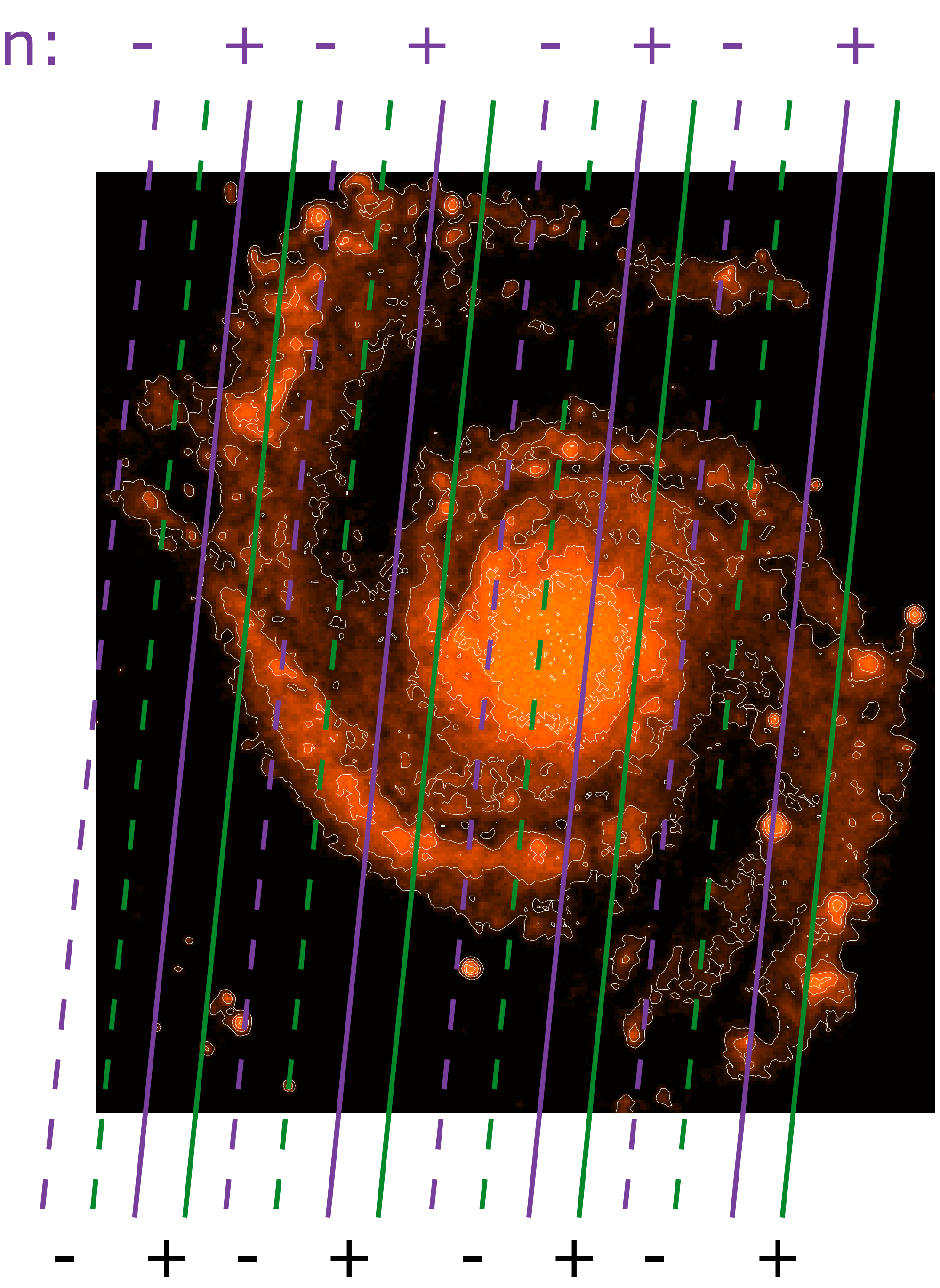
$$R_s = \int\int I_\nu(\hat{s})\sin(2\pi\nu\vec{b}\cdot\hat{s}/c)\,d\Omega = \int\int I_O(\hat{s})\sin(2\pi\nu\vec{b}\cdot\hat{s}/c)\,d\Omega$$

because the integral of the product of an odd and an odd function is non-zero!

- Remember that a sin pattern can be obtained by shifting a cosine pattern by one quarter of a period .....

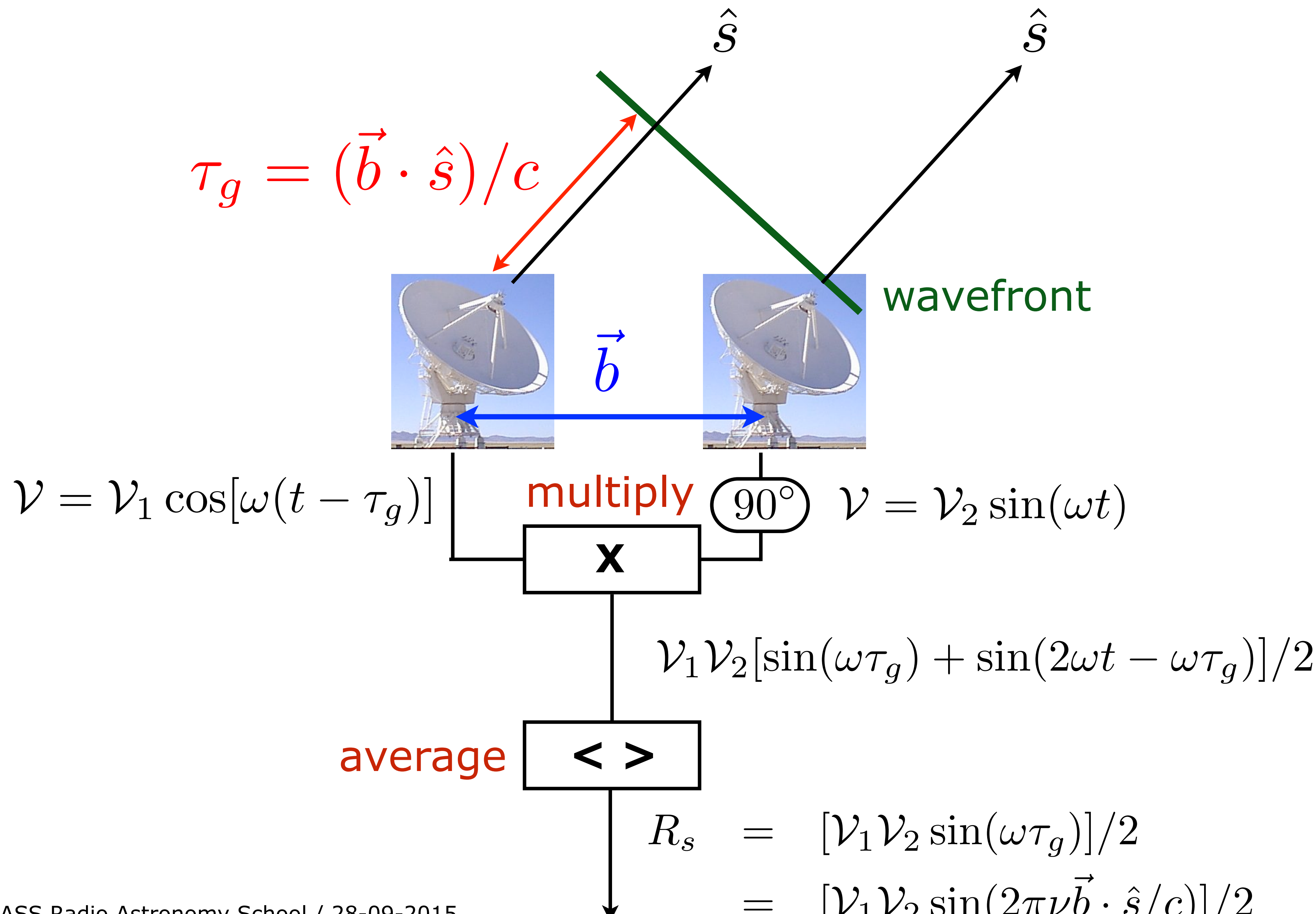
Even and odd coherence patterns

- To obtain the odd part, shift the cos pattern by 1/4 period to get a sin pattern



How to make a sin correlator

- Just add a pi/2 phase shift to one of the signal paths...



Combine to form complex visibilities

- We define a complex visibility to be

$$V = R_c - iR_s = Ae^{-i\phi}$$

with

$$A = (R_c^2 + R_s^2)^{1/2}$$ Amplitude

$$\phi = \tan^{-1}\left(\frac{R_s}{R_c}\right)$$ Phase

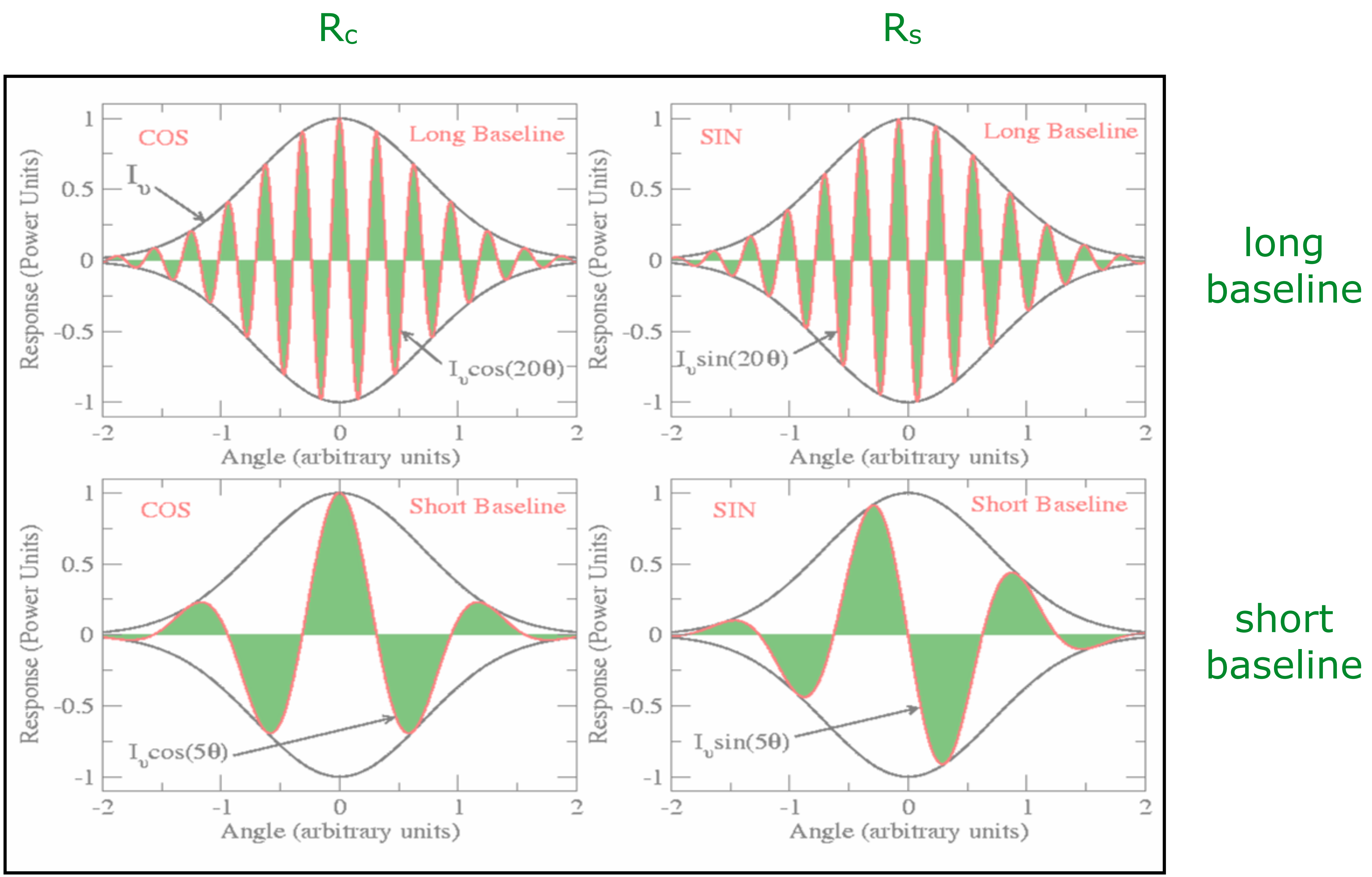
and from this we have

$$V(\vec{b}) = R_c - iR_s = \int\int I_\nu(\hat{s})\, e^{-2\pi i\nu\vec{b}\cdot\hat{s}/c}\, d\Omega$$

- This gives us a direct relationship between the source intensity distribution and the observed visibilities
- This can be **inverted** to get I(s) from V(b)

Visibilities from intensity distributions

- The intensity is in black; the fringes are in red.
  The visibility is in green (sum of the green shaded areas)



More about visibilities

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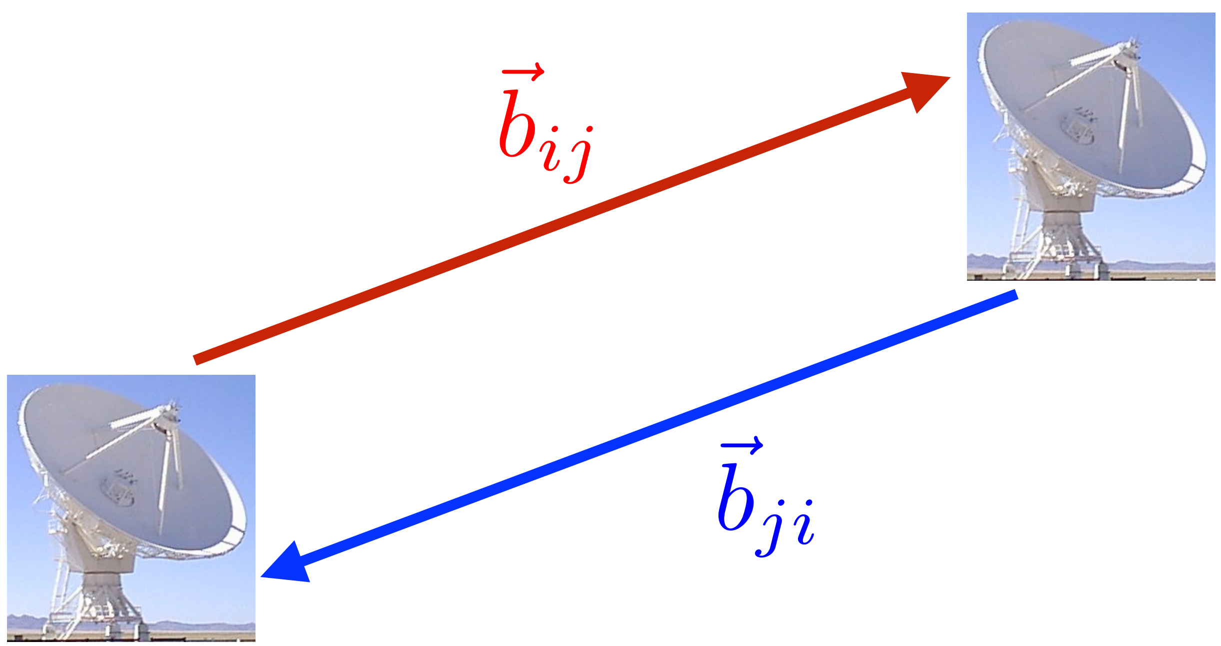
- The visibility is a function of the baseline and the source (intensity) structure
- Note that it is NOT dependent on the absolute positions of the individual antennas: we only care about the distances between elements.
- Visibilities are Hermitian, in other words:

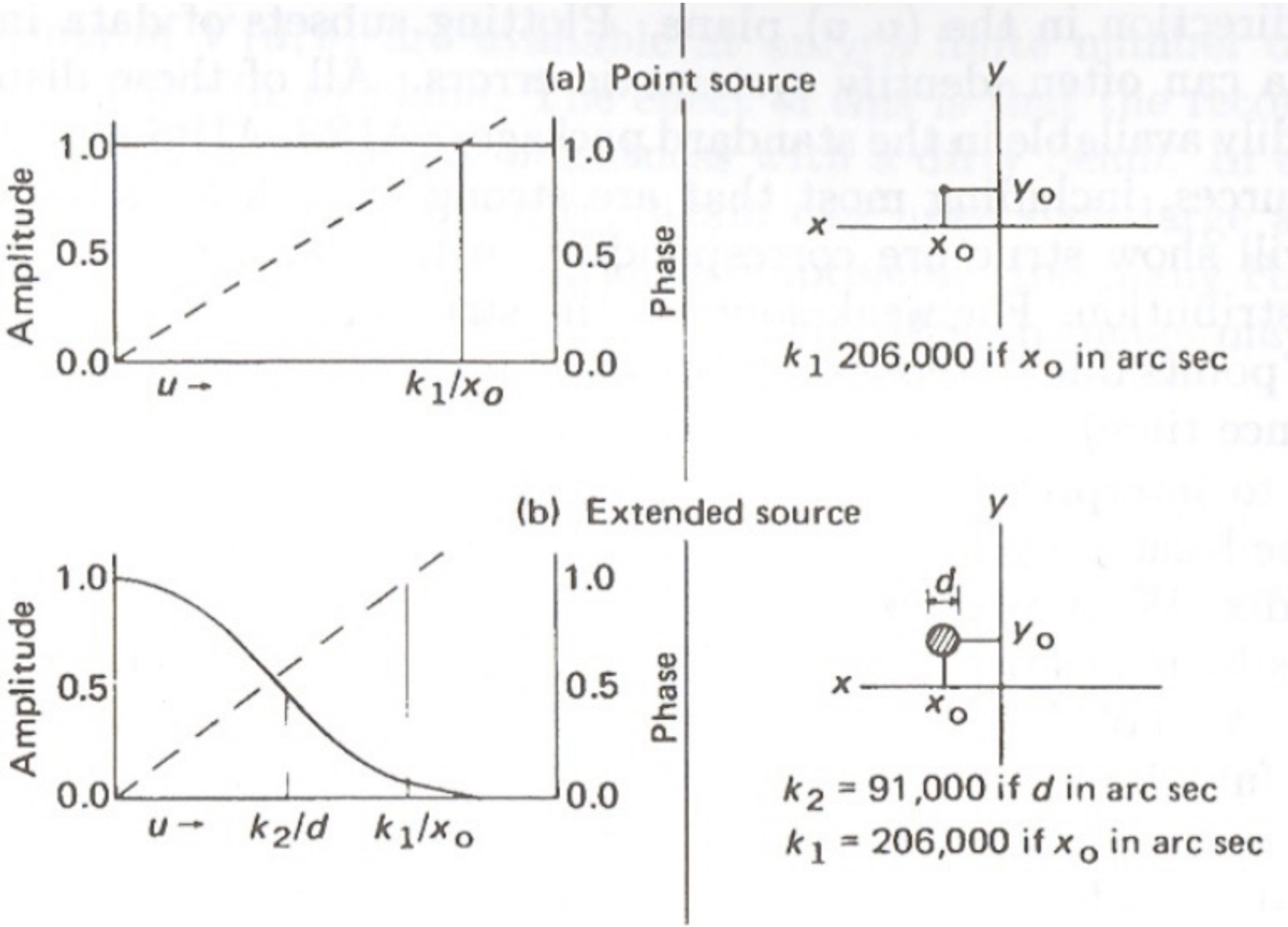


$$V(\vec{b}_{ij}) = V^*(\vec{b}_{ji})$$

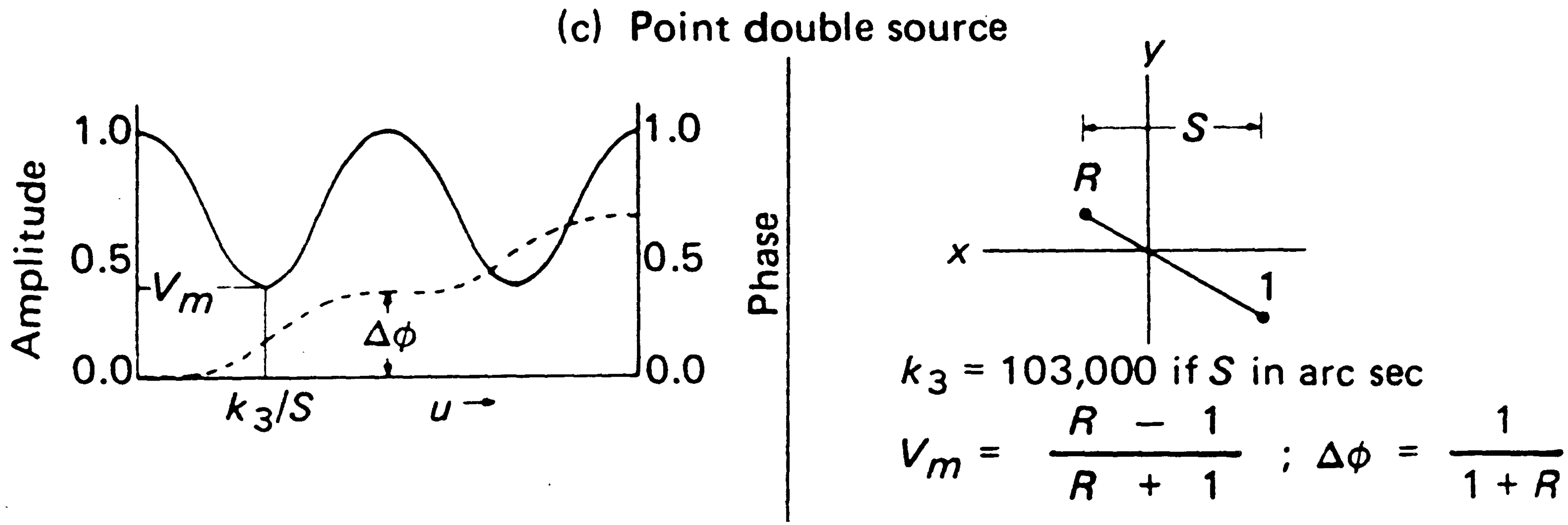
This is a consequence of the fact that the source brightness distribution is real.

- Every measurement of the source with a given baseline length and orientation gives one measure of the visibility.
- If we sample enough of the visibility function, we can make a reasonable estimate of the source brightness distribution.

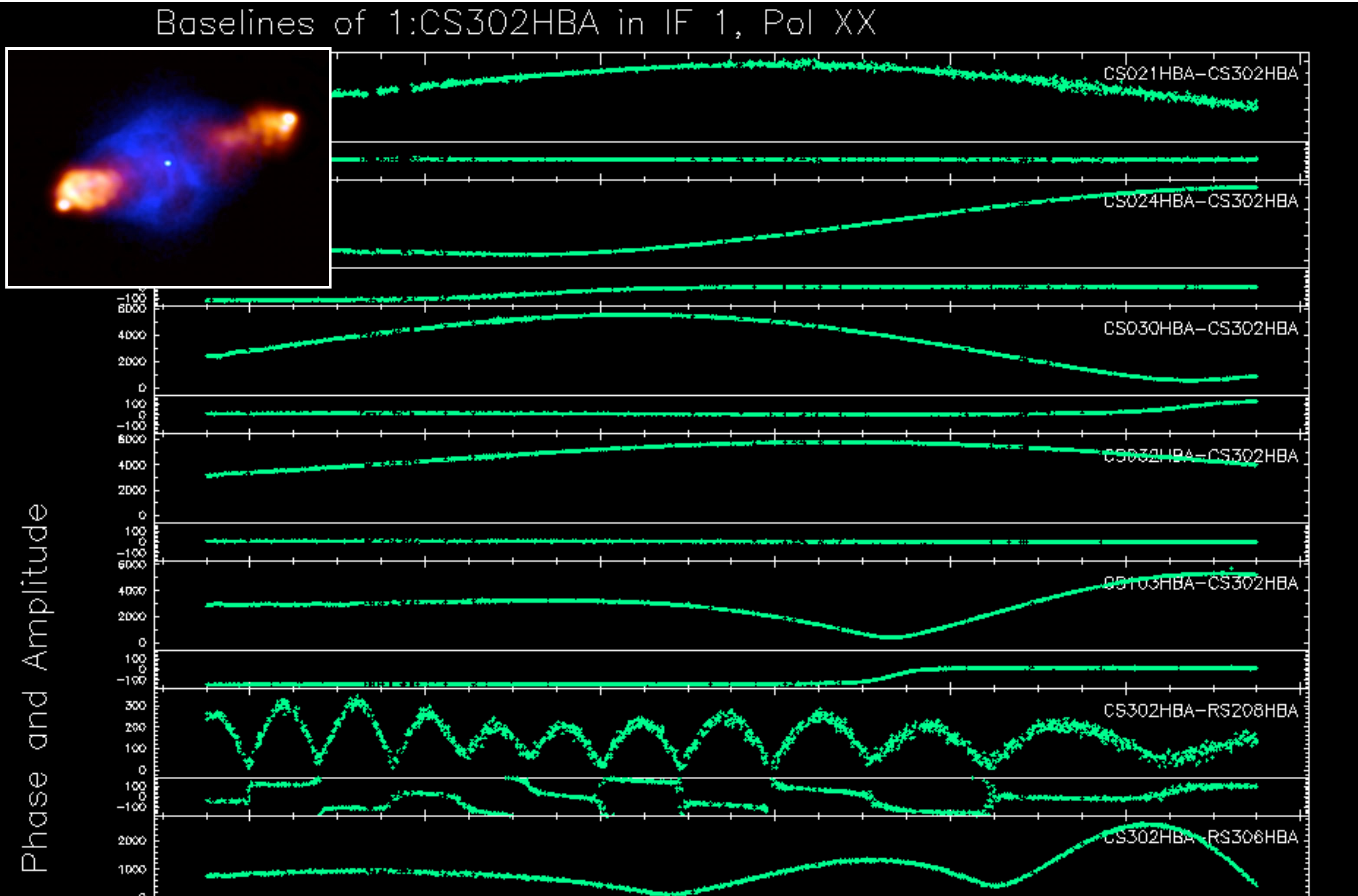
Single source



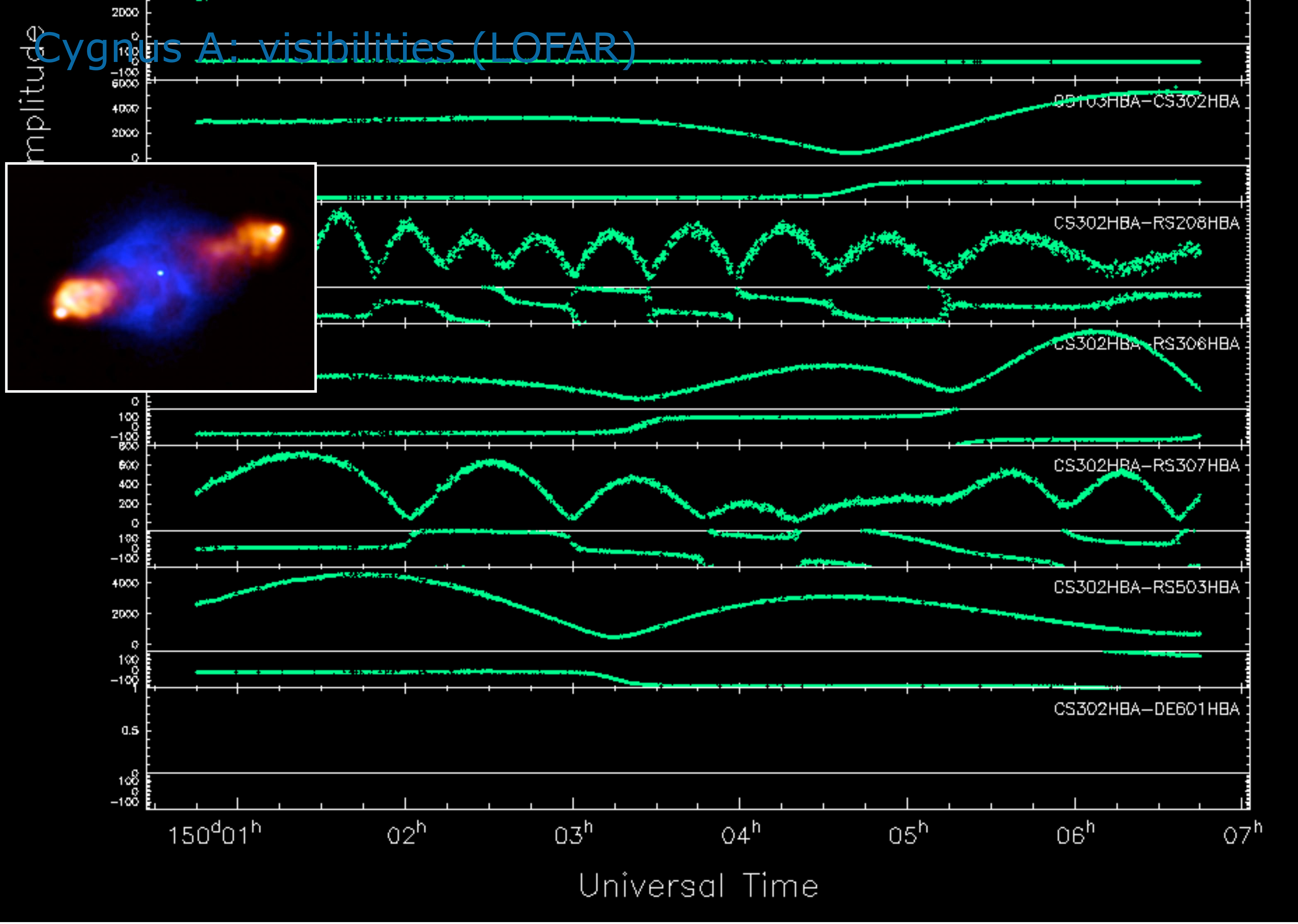
Multiple sources



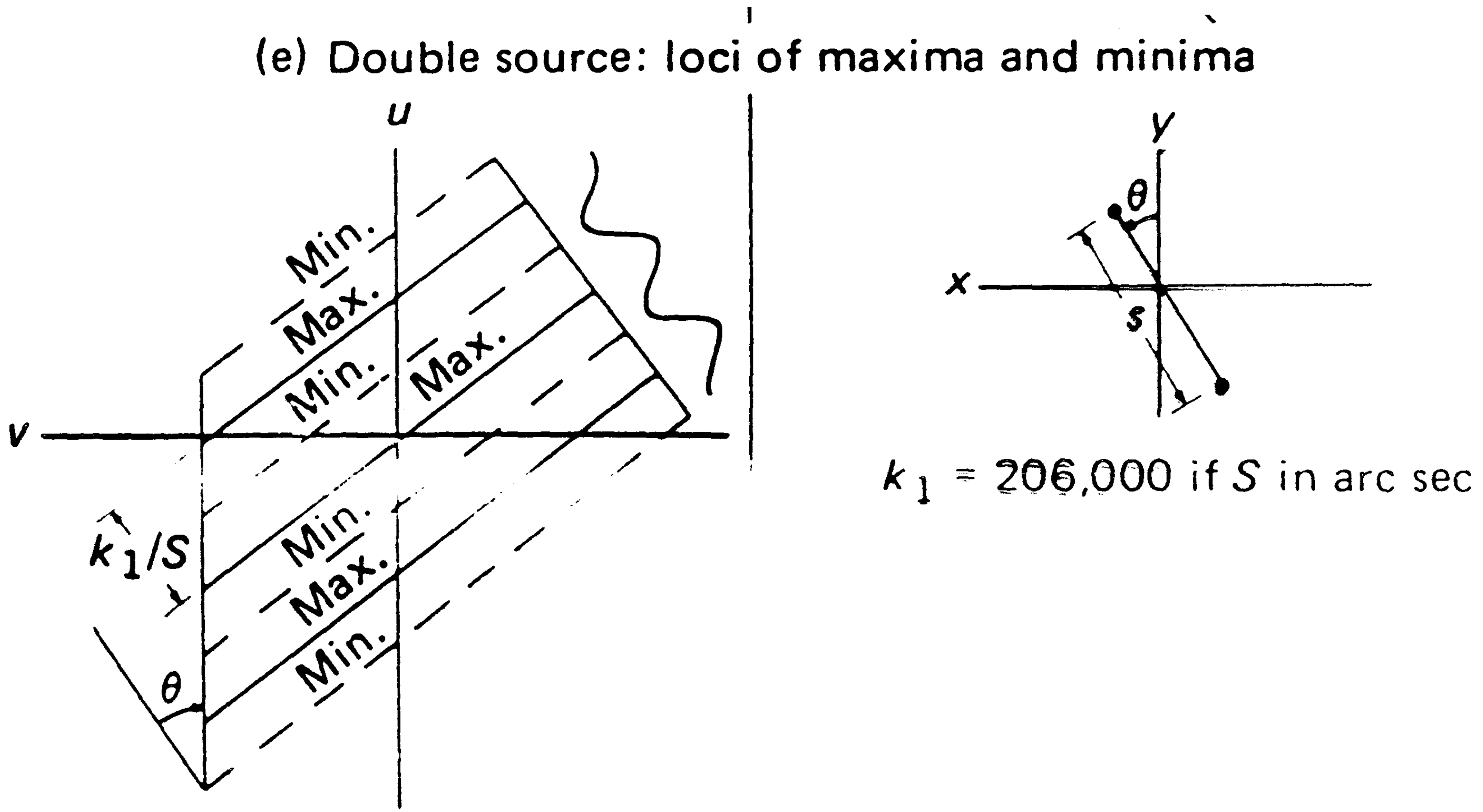
Cygnus A: visibilities (LOFAR)



Cygnus A: visibilities (LOFAR)



2-D FT of 2 sources

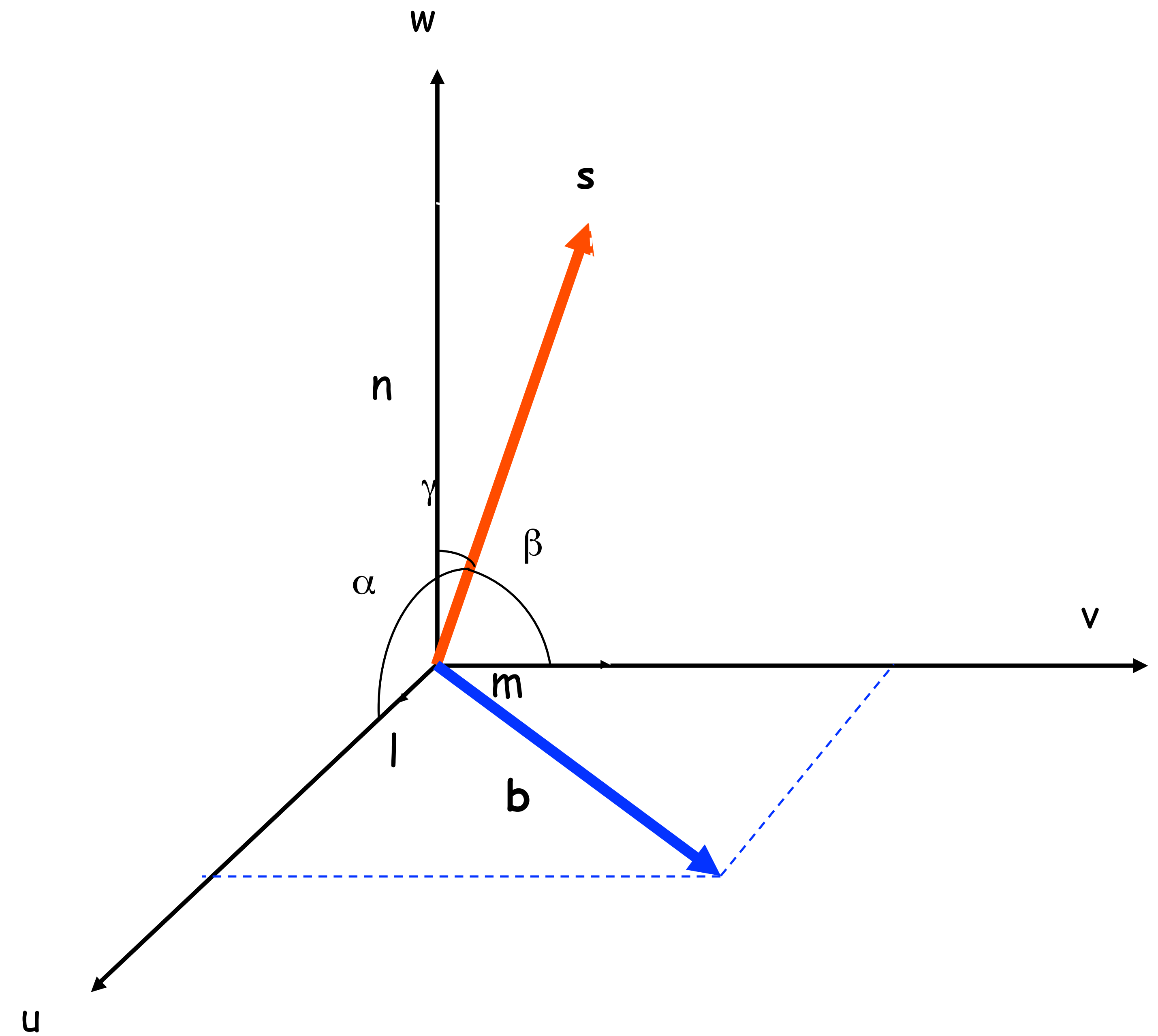


This is the **visibility function**
of a double point source
on the **(u,v) plane**

Imaging Geometry

- The unit direction vector s is defined by its projections on the (u,v,w) axes. These components are called the direction cosines.

The baseline vector b is specified by its coordinates (u,v,w) which are measured in wavelengths



Interferometers with 2-D geometry

- Which interferometers can use a 2-D geometry?
  - Those whose baselines, over time, lie on a plane (any plane).
    - This includes all E-W interferometers. For these, the w-coordinate points to the NCP.
      - Westerbork Synthesis Radio Telescope (WSRT)
      - Australia Telescope Compact Array (ATCA)
  - Any coplanar array, at a single instance of time.
      - Very Large Array (VLA) [snapshot mode]
      - Giant Metrewave Radio Telescope (GMRT) [snapshot mode]
- Is there a downside of this geometry?
  - Full resolution is obtained only for observations that are in the w-direction. Observations at other directions lose resolution.
    - E-W interferometers have no N-S resolution for observations at the celestial equator
    - A VLA snapshot of a zenith source has no resolution along the elevation dimension for objects on the horizon.

Interferometers with 3-D geometry

- More generally we have a 3-D measurement volume
  - e.g. if the interferometer layout and/or the field of view cannot be approximated as flat surfaces. Then we need a slightly different coordinate system.
  - The full expression, taking into account the geometry, is

$$V(u,v,w) = \int\int \frac{I(l,m)}{\sqrt{1-l^2-m^2}} e^{-2\pi i(ul+vm+wn)} \, dl \, dm$$

NOTE, this is not a 3-D Fourier transform.

- If we orient the coordinate system so that the w-axis points to the center of the region of interest (so that u points east and v points north) and make use of the small angle approximation,

$$V(u,v) = e^{2\pi i w} \int\int \frac{I(l,m)}{\sqrt{1-l^2-m^2}} e^{-2\pi i(ul+vm)} \, dl \, dm$$

- This also breaks down due to the small angle approximation (see lecture on advanced imaging techniques to understand how we proceed from there)

3-D coordinate system

- The w coordinate points to the source (instead of NCP in the 2D picture), u points to the east, and v toward the NCP.
  The direction cosines l and m increase to the east and the north respectively

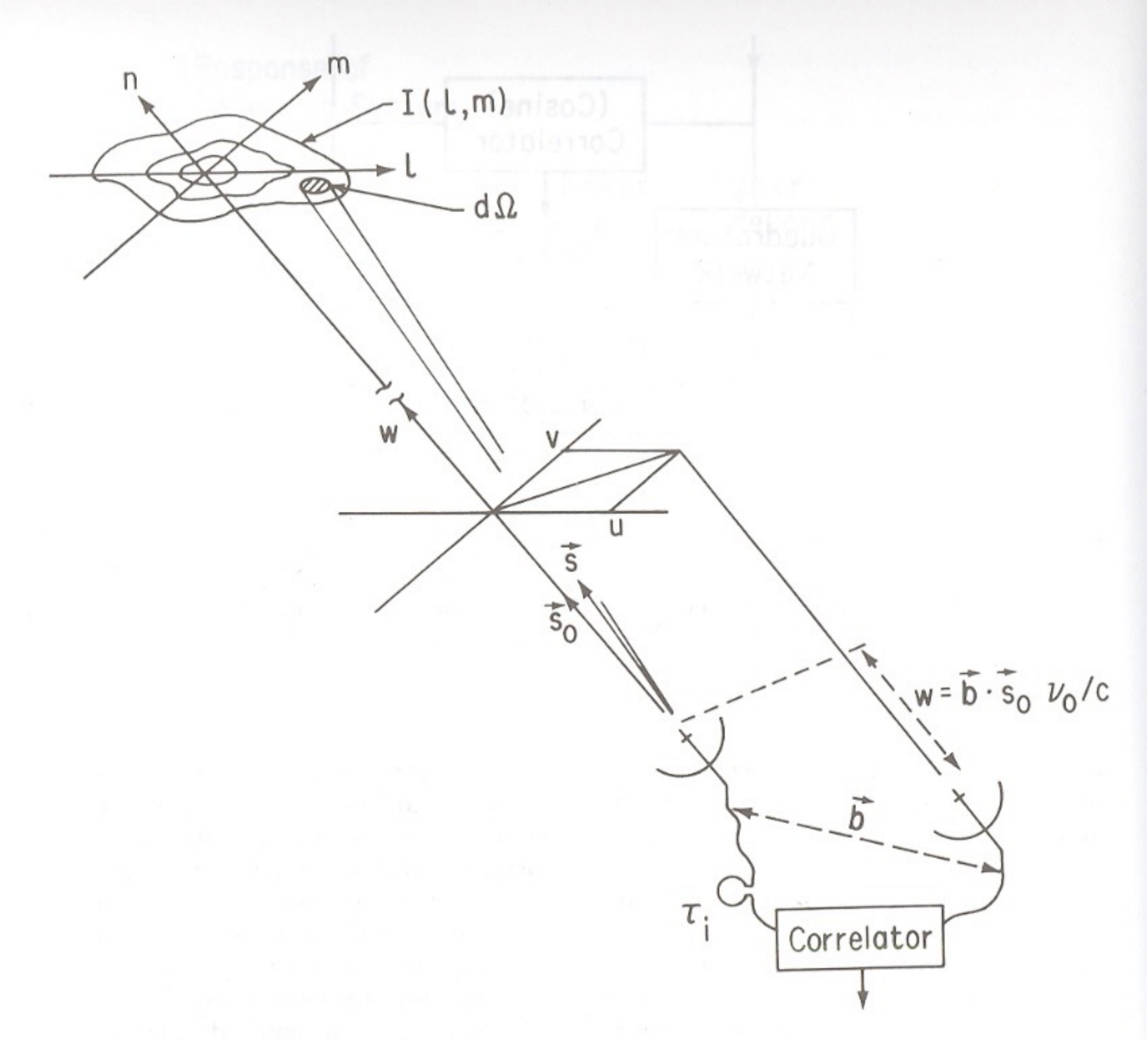
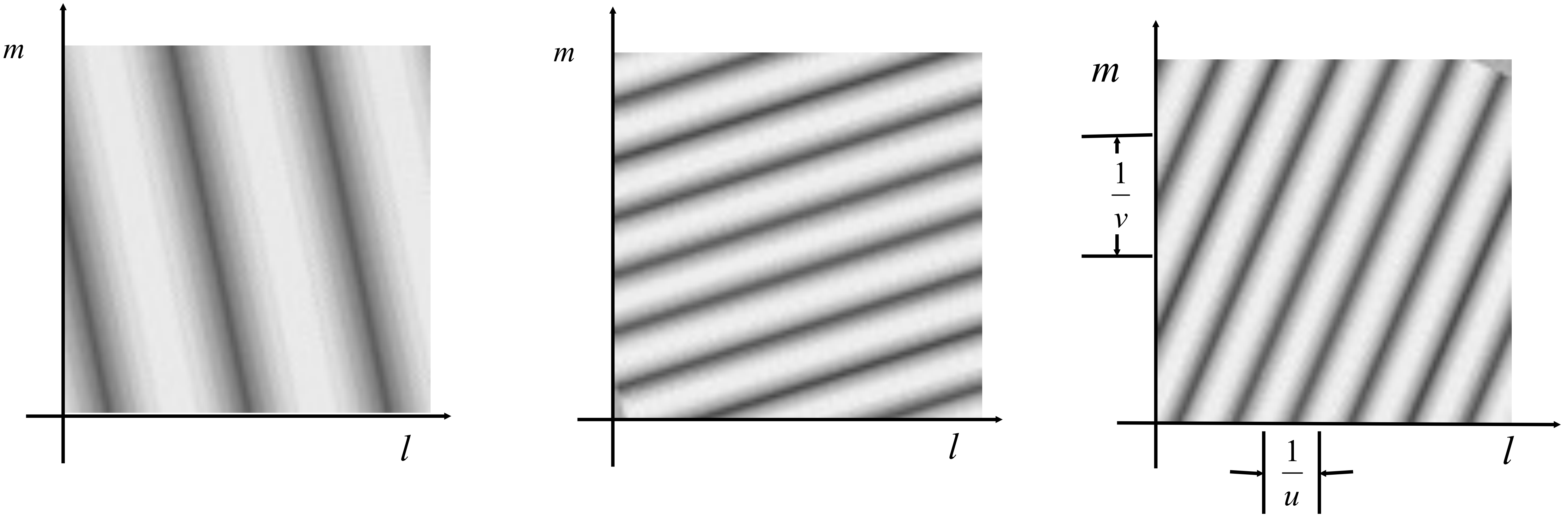


Image synthesis

- Intensity distribution built up out of fringes like these:



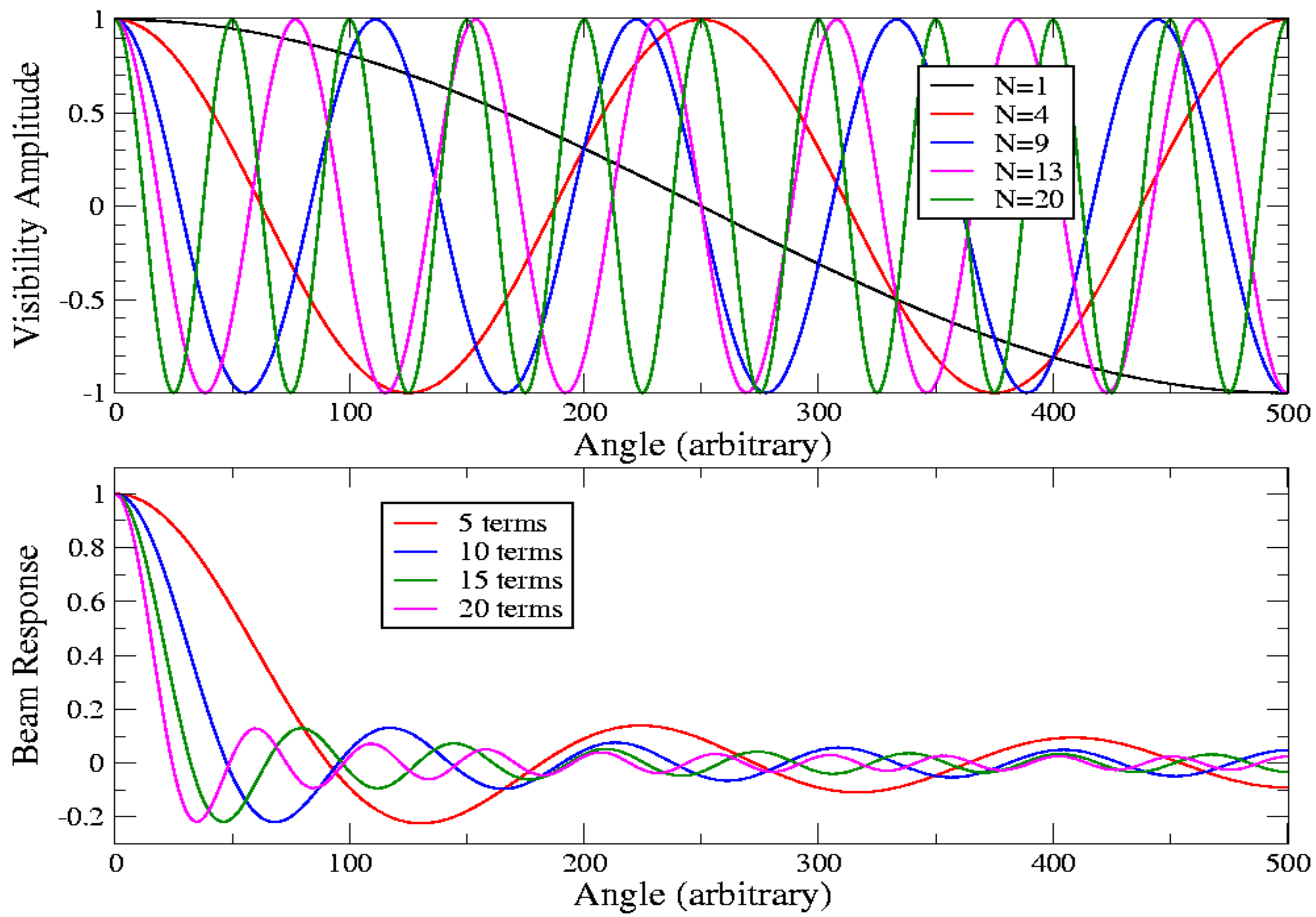
- Spatial frequency / angle determined by location of visibility in u,v plane.
- Brightness of the fringe determined by the visibility amplitude,
  and its shift relative to phase center is determined by the visibility phase.

Amplitude ⇔ Intensity

Phase ⇔ Position

1-D example: pointsource

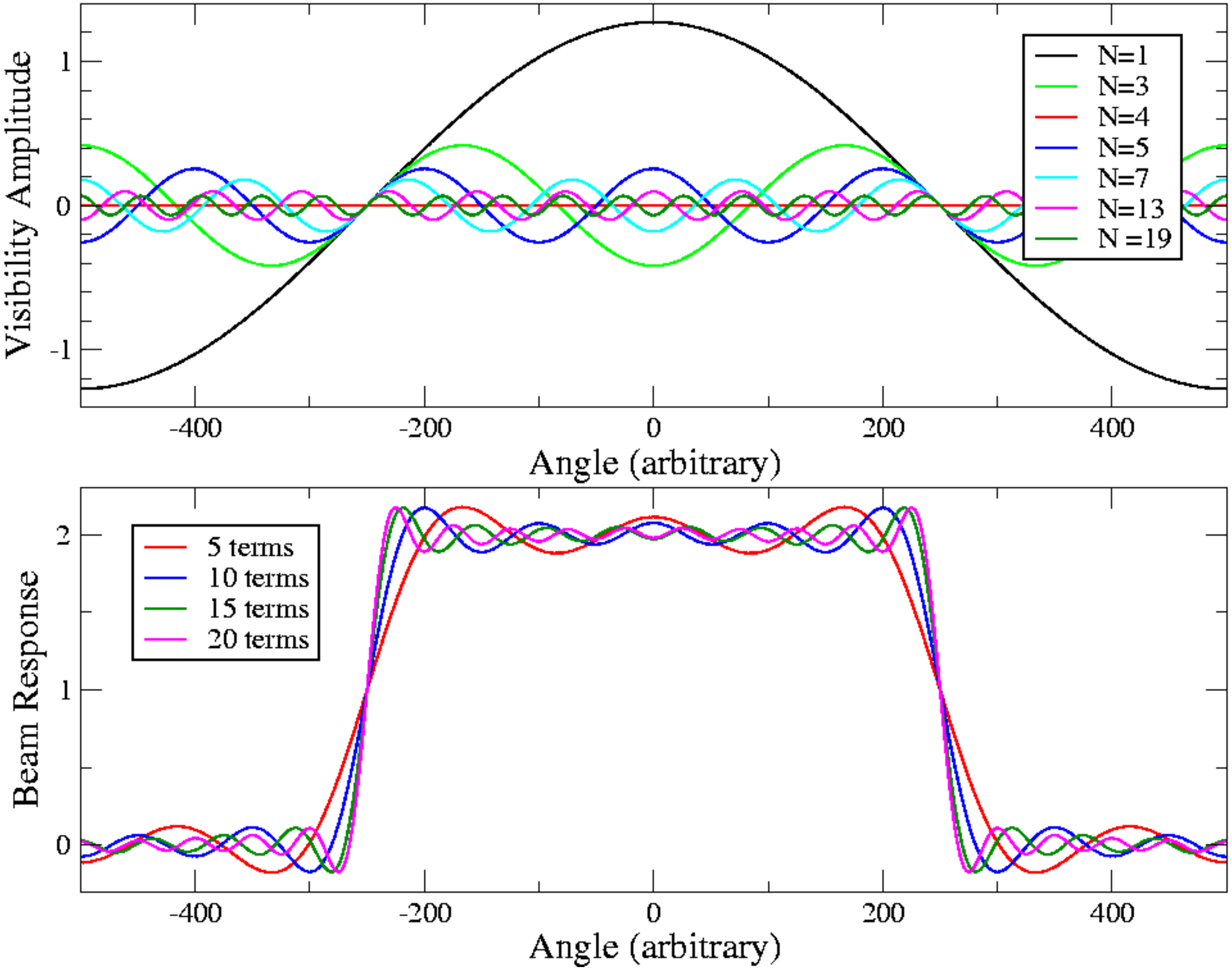
- For a unit point source at the phase center,
  - all visibility amplitudes are 1 Jy
  - all phases are zero
- The lower panel shows the response when visibilities from 21 equally spaced (and progressively longer) baselines are added.
- The individual visibilities are shown in the top panel. Their (incremental) sums are shown in the lower panel.
- In perpendicular direction, no resolution



Example 2: square source

- For a centered square object, the visibility amplitudes decline with increasing baseline, and the phases are all zero or 180º.

- Again, 21 baselines are included.



Exploit Earth's rotation to sample uv plane

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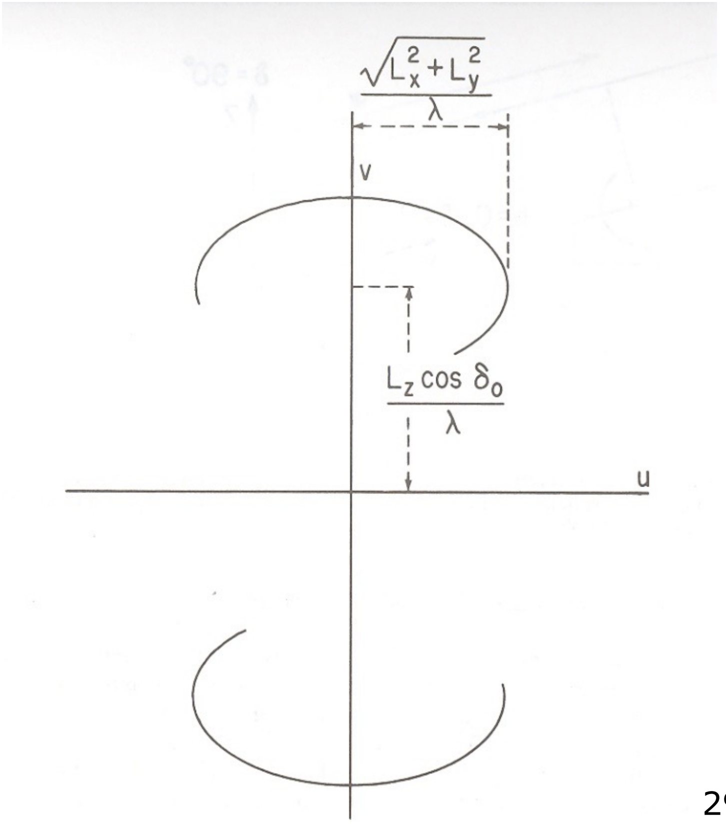
- u,v,w coordinates:
  X pointing to ha=0h, dec=0°
  Y pointing to ha=-6h, dec=0°
  Z pointing to dec=90°
- L_x, L_y, and L_z are coordinate differences of 2 antennas, h is the hour angle, δ is the declination

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} \sin h & \cos h & 0 \\ -\sin\delta\cos h & \sin\delta\sin h & \cos\delta \\ \cos\delta\cos h & -\cos\delta\sin h & \sin\delta \end{pmatrix} \begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix}$$

so that

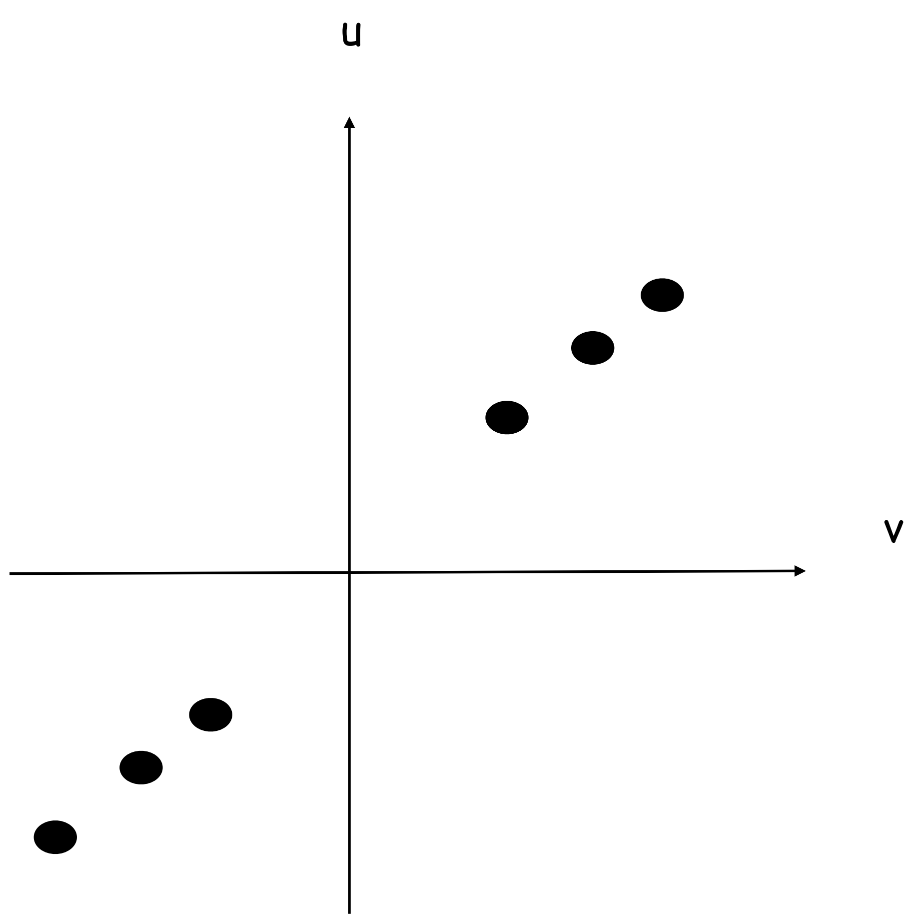
$$u^2 + \left(\frac{v - (L_z/\lambda)\cos\delta}{\sin\delta}\right)^2 = \frac{L_x^2 + L_y^2}{\lambda^2}$$

As Earth rotates, baselines describe an ellipse
For EW baselines, L_z=L_x=0
-> concentric, coplanar ellipses

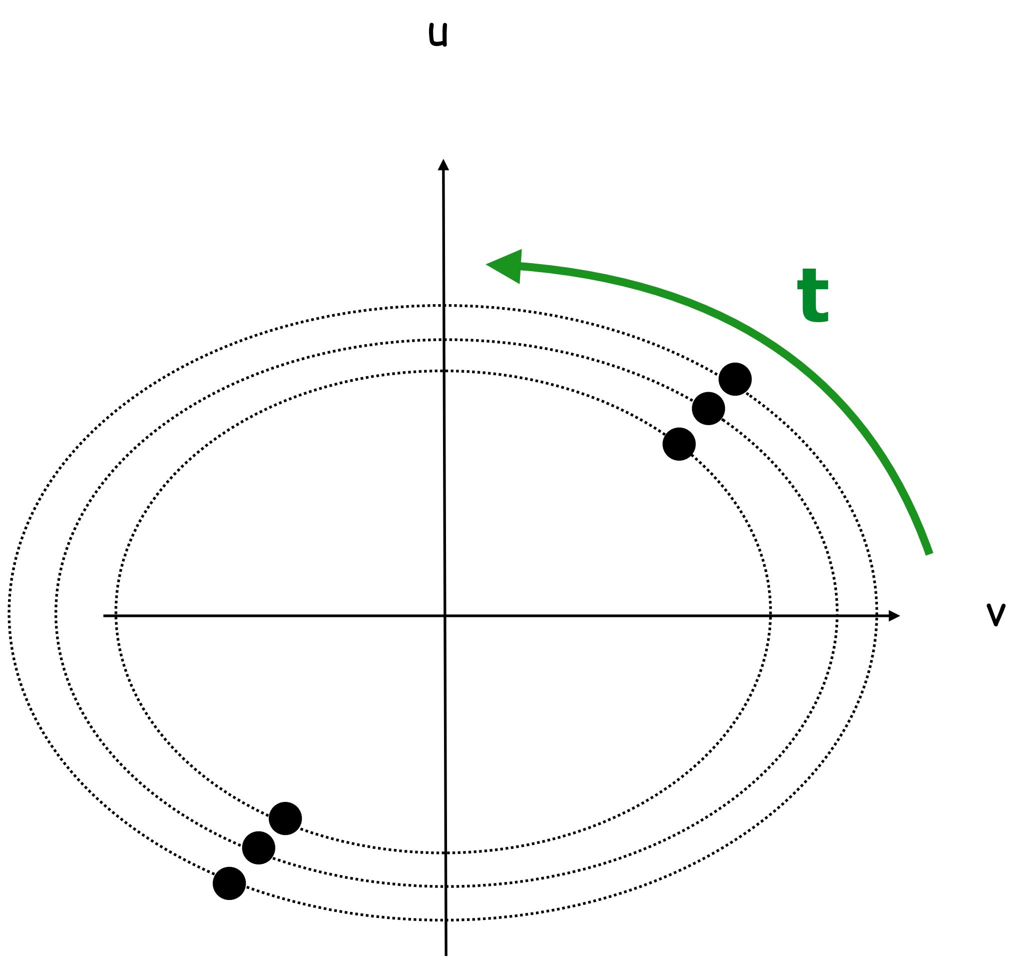


Exploit Earth's rotation to sample uv plane

- At one instant, EW baselines sample a line in the uv plane
- After 12 hours, they fill out **ellipses** in the uv plane
- Bandwidth can be exploited to fill uv plane radially (addressed in later lectures)



Instantaneous uv coverage

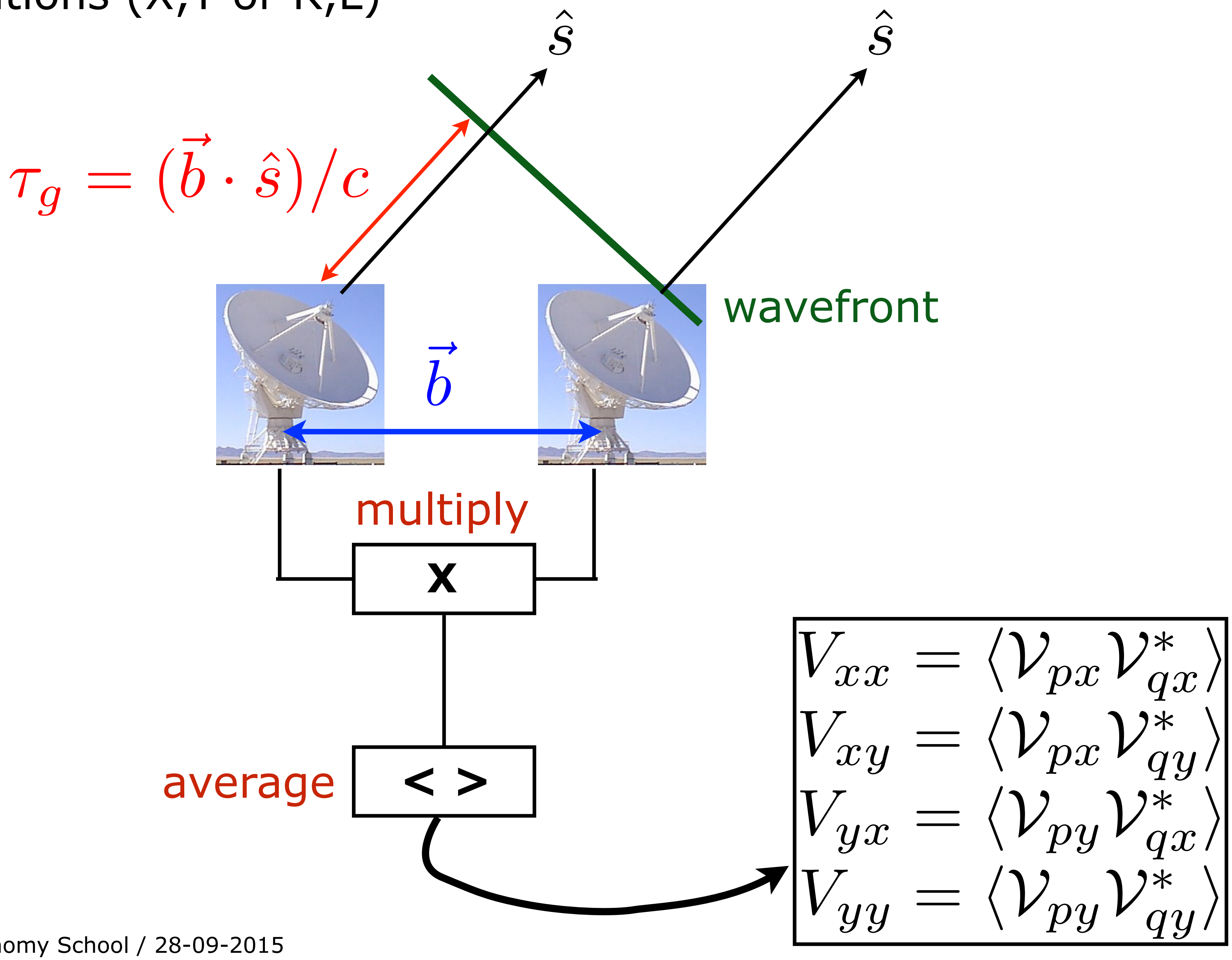


Earth rotation aperture synthesis produces filled uv coverage

The Measurement Equation

- The measurement equation (Hamaker, Bregman & Sault) is a matrix formalism for expressing the polarimetric response of an interferometer

Here: p,q are antenna indices
x,y are polarizations (X,Y or R,L)



The Measurement Equation

- The measurement equation (Hamaker, Bregman & Sault) is a matrix formalism for expressing the polarimetric response of an interferometer

$$
\begin{aligned}
V_{xx} &= \langle \mathcal{V}_{px} \mathcal{V}_{qx}^* \rangle \\
V_{xy} &= \langle \mathcal{V}_{px} \mathcal{V}_{qy}^* \rangle \\
V_{yx} &= \langle \mathcal{V}_{py} \mathcal{V}_{qx}^* \rangle \\
V_{yy} &= \langle \mathcal{V}_{py} \mathcal{V}_{qy}^* \rangle
\end{aligned}
$$

can be more easily and elegantly expressed in matrix form, as:

$$
\mathbf{V}_{pq} = \langle \vec{\mathcal{V}}_p \vec{\mathcal{V}}_q^\dagger \rangle = \langle \begin{pmatrix} \mathcal{V}_{px} \\ \mathcal{V}_{py} \end{pmatrix} \begin{pmatrix} \mathcal{V}_{qx}^* & \mathcal{V}_{qy}^* \end{pmatrix} \rangle = \begin{pmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{pmatrix}
$$

The Measurement Equation

- The measurement equation (Hamaker, Bregman & Sault) is a matrix formalism for expressing the polarimetric response of an interferometer
- Introducing the coherency matrix **C**, which describes the intensity distribution:

$$\mathbf{C}_{pq} = \begin{pmatrix} I+Q & U+iV \\ U-iV & I-Q \end{pmatrix} = \langle \vec{E}\vec{E}^{\dagger} \rangle$$

- and the "Jones matrix" J which contains all of the information about what happens to the signal, from the source to the correlator,

$$\mathcal{V}_p = \mathbf{J}_p \vec{E} \qquad \mathcal{V}_q = \mathbf{J}_q \vec{E}$$

- then with a bit of math we can write down the measurement equation:

$$\mathbf{V}_{pq} = \mathbf{J}_p \mathbf{C}_{pq} \mathbf{J}_q^{\dagger} = \mathbf{J}_{pq} \mathbf{C}_{pq}$$

The Measurement Equation

- The Jones matrices contain all of the stuff that we have to calibrate

$$\mathbf{V}_{pq} = \mathbf{J}_{pq}\mathbf{C}_{pq}$$

- For example,
    - G: the (complex) antenna gain
    - B: bandpass
    - F: Faraday rotation
    - E: antenna response pattern

$$\mathbf{V}_{pq} = \mathbf{M}_{pq}\mathbf{B}_{pq}\mathbf{G}_{pq}\mathbf{D}_{pq}\mathbf{E}_{pq}\mathbf{P}_{pq}\mathbf{T}_{pq}\mathbf{F}_{pq}\mathbf{C}_{pq}$$

The Measurement Equation

- So what?

$$\mathbf{V}_{pq} = \mathbf{J}_{pq}\mathbf{C}_{pq}$$

- The measurement equation makes it more straightforward to handle polarization calibration, and direction dependent effects
- Explicit separation of dependencies (for example, G(t) and B(ν) are the antenna gain and bandpass)
- Necessary in order to understand advanced calibration!

- We'll return to this in the Calibration lecture tomorrow

Key concepts

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2. Correlation takes place by multiplying and time-averaging antenna voltages
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