KLT for transient signal analysis

The Labyrinth of the Unexpected:
unforeseen treasures in impossible regions of phase space
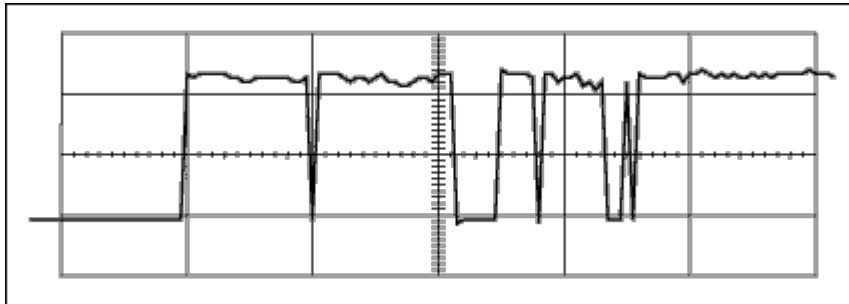
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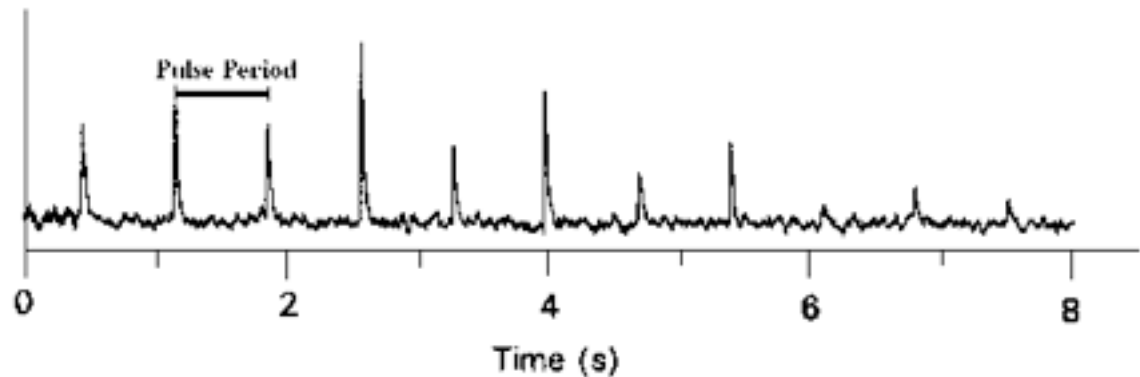# KLT for transient signal analysis

Qualitatively definition of transient signals

- Signals that
  - last for a very short period of time
  - are born from a sudden change in amplitude

  are referred to as TRANSIENT signals

Switch bounces

Recording of the Vela pulsar (PSR B0833-45) made at the Green Bank 140-ft telescope of the U.S. National Radio Astronomy Observatory, Charlottesville VA, in September, 1970.

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Mathematical definition of transient signals

- A qualitatively definition is as follows: a signal is said to contain a transient signal when it is short in time and its slope/amplitude abruptly changes with respect to the previous instant in time

- Because of the uncertainty principle between time and frequency, the narrower is a signal in time, the broader is the frequency range to take into account to describe the signal. Therefore

- A mathematically convenient definition is as follows: a signal is said to contain a *transient* whenever its Fourier expansion requires an *infinite* number of sinusoids. Thus, waveform discontinuities are transients, as are discontinuities in the waveform slope, curvature, etc.

- Thus, waveform discontinuities are transients, as are discontinuities in the waveform slope, curvature, etc. Any fixed sum of sinusoids, on the other hand, is a steady-state signal.

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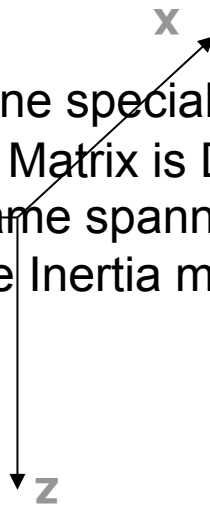
Transient signals in SETI as well

- Level 1: **Piggyback SETI at 1.420 GHz by FFT.**

- Level 2: **Broad-band SETI by KLT**

- Level 3: **Targeted Searches**

- Level 4: **Leakage Searches**

- Level 5: **Entanglement & Encrypted SETI (?).**

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Eigenfunctions

Example (a Newtonian analogy): consider a solid object, like a BOOK, described by its INERTIA MATRIX.

Then there exist only one special reference frame where the Inertia Matrix is DIAGONAL. This is the reference frame spanned by the EIGENVECTORS of the Inertia matrix.

$$I' \equiv \begin{pmatrix} I_{x_x}^{\hat{x}} & I_0^y & I_0^{\bar{z}} \\ I_0^x & I_{x_y}^{\hat{y}} & I_0^{\hat{x}} \\ I_0^x & I_0^{\hat{x}} & I_{x_z}^{\hat{x}z} \end{pmatrix}$$

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Mathematics of the KLT

- $X(t)$ is a stochastic process and it can be expanded into an infinite series on a finite support, called the Karhunen-Loéve Transform (KLT)

$$X(t) = \sum_{n=1}^{\infty} Z_n \, \phi_n(t) \qquad 0 \le t \le T$$

- $\phi_n(t)$ are orthonormalized functions of the time: $\displaystyle\int_0^T \phi_m(t)\,\phi_n(t)\,dt = \delta_{mn}$

- Z_n are random variables, not depending on time: $\langle Z_m \, Z_n \rangle = \lambda_n \, \delta_{mn}$

- The KLT separates the input (signal + noise) to the radio telescope, into the stochastic part and the deterministic part in time

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It's about solving an integral equation

$$\int_0^T \langle X(t_1) X(t_2) \rangle \, \phi_n(t_1) \, dt_1 = \lambda_n \, \phi_n(t_2)$$

- This is the integral equation yielding the Karhunen-Loève's $\phi_n(t)$ eigenfunctions and corresponding eigenvalues λ_n.

- It is the usual eigenfunction of the quantum mechanics but, instead of the potential, here the operator we apply to the eigenfunctions is the autocorrelation (kernel of the transform) and, instead of a differential equation, here is an integral equation

- This is the best basis in the Hilbert space describing the (signal + noise). The KLT adapts itself to the shape of the radio telescope input (signal + noise) by adopting, as a reference frame, the one spanned by the eigenfunctions of the autocorrelation. And this is turns out to be just a LINEAR transformation of coordinates in the Hilbert space

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KLT filtering

- There's no degeneracy: each eigenvalue correspond to only one eigenfunction

- The eigenvalues turn out to be the variances of the random variables Z_n , that is $\sigma_{Z_n}^2 = \lambda_n = \langle Z_n^2 \rangle > 0$.

- Since $\langle Z_n \rangle = 0$, we can SORT in descending order of magnitude both the eigenvalues and the corresponding eigenfunctions. Then, if we decide to consider only the first few eigenfunctions as the "bulk" of the signal

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KLT filtering vs. FFT

	KLT		FFT	
↑	Works well for both wide and narrow band signals		Practically true for narrow band signals only (Nyquist criterion)	↓
↑	Works for both stationary and non-stationary input stochastic processes		Works OK for stationary input stochastic processes only (Wiener-Khinchine theorem)	↓
↑	Is defined for any finite time interval		Is plagued by the "windowing" problems and the Gibbs phenomenon	↓
↓	Needs high computational burden: no "fast" KLT		Fast algorithm FFT	↑

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Final variance theorem and Bordered Autocorrelation Method 1/2

- The Final Variance Theorem (Maccone, Proc. of Science, 2007) proves the dependency of the eigenvalues on the final instant of our measurement, *T*

$$\sum_{n=1}^{\infty} \lambda_n(T) = \int_0^T \sigma_{X(t)}^2 \, dt$$

- We can now take the derivative of both sides and get

$$\sum_{n=1}^{\infty} \frac{\partial \lambda_n(T)}{\partial T} = \sigma_{X(T)}^2$$

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Final Variance Theorem and Bordered Autocorrelation Method 2/2

- For a stationary stochastic process only

$$\sum_{n=1}^{\infty} \frac{\partial \lambda_n(T)}{\partial T} \approx \frac{\partial \lambda_1(T)}{\partial T} = \sigma_X^2 = a$$

that is, the sum of the derivative of the eigenvalues with respect to the final instant T is a constant

- In general, the behavior of the eigenvalues with respect to the final instant T can be found by increasing the size of the autocorrelation matrix ad calculating, at every step, the set of eigenvalues (Bordered Autocorrelation Method)

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Bordered Autocorrelation Method, an analytical sample

- Let's analyze a single harmonic frequency buried in white noise

$$S(t) = W(t) + \mu \sin(\omega t) \quad (\mu << 1) \quad (0 \leq t \leq T)$$

- It can be proved that the eigenvalues $\lambda_n(T)$ of the stochastic process $S(t)$

$$\lambda_n(T) = \frac{8 \pi^2 \mu^2 n^2 T}{\left(\omega^2 T^2 - 4 \pi^2 n^2\right)^2} \cdot \left[\frac{1}{2} - \frac{1}{2} \cos(2 \omega T)\right]$$

and

$$\frac{\partial \lambda_n(T)}{\partial T} \approx c_1(T) \sin(2\omega T) + c_2(T) \cos(2\omega T) + c_3(T)$$

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KLT for wide band and feeble signals

- Thus, the result is that the frequency of the first derivative if the first eigenvalue with respect to the final instant *T* is twice the frequency of the signal buried in the white noise

- Using the Bordered Autocorrelation Method, it was possible to filter out a simulated harmonic signal buried in noise with a signal to noise ratio (SNR) as low as 0,005, whereas the FFT fails to retrieve it (Acta Astronautica, *The KLT to extend SETI searches to broad-band and extremely feeble signals*, Claudio Maccone)

- Using the KLT, it was possible to get power spectrum of a chirp signal with a SNR = -12dB (ESA working paper, *Karhunen-Loève Transform as an instrument to detect weak RF signals*, Arkadiusz Szumski)

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Lines for the future

- The frequency behavior of the derivative of the first eigenvalue with respect to the final instant T in case of stationary white noise was neither depending on T nor n; nevertheless, a more complicated, process could lead to such a dependence. An investigation of its behavior might give hints to solve the folding problem given the signal you are looking at

- Given their ability to model signals that are feeble and wideband, the KLT eigenfunctions might be suitable to deep learning processes at detecting SETI signals or transient signals. Principal component analysis and Deep learning processes are already used for speech and image recognition

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Thank you