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KLT of radio signals from relativistic spaceships in hyperbolic motion

19.1 INTRODUCTION

A spaceship, traveling at a constant acceleration g in its own reference frame, exemplifies the relativistic interstellar flight. If a Gaussian noise (Brownian motion) is emitted in units of the spaceship's proper time, it undergoes a time rescaling when measured in units of the coordinate time. This noise is studied in this chapter in terms of its KL expansion. All topics discussed in this chapter were first published by the author between 1988 and 1990 [1, 2].

19.2 HYPERBOLIC MOTION

A classical topic in special relativity is the so-called *hyperbolic motion*, first considered by Minkowski in 1908 [3], and discussed in most textbooks (see [4, p. 41]).[1] Spaceflight did not exist in the time of Minkowski, so he believed that his formulas about the hyperbolic motion could only be applied to the physics of elementary particles then known to exist, such as electrons. Here, however, we shall give the topic of hyperbolic motion a space-travel cut, in view of the applications to telecommunications that will be made in the rest of this book.

Imagine a spacecraft traveling faster and faster with respect to its own reference frame, so that the crew experience a constant acceleration that, for their maximum comfort, we assume numerically equal to $g = 9.8\ \text{m/s}^2$. The longitudinal force (see [5, p. 205]) is

$$f_{\parallel} = \left[1 - \frac{v^2(t)}{c^2}\right]^{-\frac{3}{2}} m \frac{dv(t)}{dt} \tag{19.1}$$

[1] The adjective "hyperbolic" refers to the fact that the $x(t)$ curve in the (x, t) plane is a hyperbola—given by Equation (13.24)—and that hyperbolic functions are used in the analysis.

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and so we must find the unknown $v(t)$ in the differential equation

$$\left[1 - \frac{v^2(t)}{c^2}\right]^{-\frac{3}{2}} m \frac{dv(t)}{dt} = mg. \tag{19.2}$$

Separating the variables, and setting $v(t) = c \sin \Omega(t)$, one easily finds

$$\Omega(t) = \arctan\left(\frac{g}{c} t\right), \tag{19.3}$$

whence

$$v(t) = c \sin\left[\arctan\left(\frac{g}{c} t\right)\right] \tag{19.4}$$

but

$$\sin[\arctan x] = \sqrt{\frac{x^2}{1 + x^2}} \tag{19.5}$$

so that the velocity $v(t)$ in (11.16) is given by

$$v(t) = \frac{gt}{\sqrt{1 + \left(\frac{g}{c} t\right)^2}}. \tag{19.6}$$

Note that as $t \to \infty$, (19.6) gives $v(t) \to c$, as one would expect. The function $f(t)$ for the hyperbolic motion is then found from (11.16) and (19.6)

$$f(t) = \frac{1}{\left[1 + \left(\frac{g}{c} t\right)^2\right]^{\frac{1}{4}}}. \tag{19.7}$$

Unfortunately, it is quite difficult to handle this function. For instance, its integral

$$\int \frac{dx}{[1 + x^2]^{\frac{1}{4}}} \tag{19.8}$$

can be shown to be expressed by hypergeometric functions inasmuch as it is a binomial integral, but not of an elementary type. Thus, we will not attempt to study (19.7) directly, but shall consider its asymptotic expansion in Section 19.4.

A few more results, however, can still be derived from (19.7). In fact, one has (see [6, p. 86])

$$\begin{aligned} \tau(t) = \int_0^t f^2(s)\, ds = \int_0^t \frac{ds}{\sqrt{1 + \left(\frac{g}{c} s\right)^2}} &= \frac{c}{g} \operatorname{arcsinh}\left(\frac{g}{c} t\right) \\ &= \frac{c}{g} \ln\left[\frac{g}{c} t + \sqrt{1 + \left(\frac{g}{c} t\right)^2}\right]. \end{aligned} \tag{19.9}$$

Thus, the time-rescaled Brownian motion corresponding to the hyperbolic motion of special relativity is

$$X(t) = B(\tau) = B\left(\frac{c}{g}\operatorname{arcsinh}\left(\frac{g}{c}t\right)\right)$$
$$= B\left(\frac{c}{g}\ln\left[\frac{g}{c}t + \sqrt{1 + \left(\frac{g}{c}t\right)^2}\right]\right). \tag{19.10}$$

We shall simply refer to it as the hyperbolic motion.

19.3 TOTAL ENERGY OF SIGNALS FROM RELATIVISTIC SPACESHIPS IN HYPERBOLIC MOTION

In this section we shall show that it is possible (by virtue of the formulas derived in Chapter 21) to compute both the mean total energy and total energy variance of the signals emitted by relativistic spaceships in hyperbolic motion.

Let us start with the mean total energy (21.60). This, by substituting (19.9), takes the form of the definite integral

$$E\{\varepsilon\} = \int_0^T dt \int_0^t f^2(s)\, ds = \frac{c}{g}\int_0^T \operatorname{arcsinh}\left(\frac{g}{c}t\right) dt$$
$$= \frac{c^2}{g^2}\left[x \operatorname{arcsinh}(x) - \sqrt{1 + x^2}\right]_0^{\frac{g}{c}T} \tag{19.11}$$

where we make use of the substitution $(gt)/c = x$ and of [6, p. 88, entry 4.6.43]. Thus, the mean total energy of the hyperbolic motion is

$$E\{\varepsilon\} = \frac{c^2}{g^2}\left[\frac{gT}{c}\operatorname{arcsinh}\left(\frac{gT}{c}\right) - \sqrt{1 + \left(\frac{gT}{c}\right)^2} + 1\right]. \tag{19.12}$$

It is also possible to derive a closed-form expression for the total energy variance starting from (21.62) and (19.9), but the calculations are more involved. To this end, let us first note that

$$\int \operatorname{arcsinh}^2(s)\, ds = s \operatorname{arcsinh}^2(s) - 2\sqrt{1 + s^2}\operatorname{arcsinh}(s) + 2s + C \tag{19.13}$$

This result can be used to prove the more complicated expression

$$\int x \operatorname{arcsinh}^2(x)\, dx = \tfrac{1}{2}x^2 \operatorname{arcsinh}^2(x) - x\sqrt{1 + x^2}\operatorname{arcsinh}(x)$$
$$+ \frac{x^2}{2} + \int \sqrt{1 + x^2}\operatorname{arcsinh}(x)\, dx + C. \tag{19.14}$$

This leads us to compute a further integral

$$\int \sqrt{1+x^2}\,\text{arcsinh}(x)\,dx = \tfrac{1}{4}[(2\sqrt{1+x^2}\,\text{arcsinh}(x) - x)x + \text{arcsinh}^2(x)] + C. \tag{19.15}$$

These preliminary results enable us to tackle σ_ε defined in (21.62) using (19.9)

$$\sigma_\varepsilon^2 = 4\int_0^T dt \int_0^t du \left[\int_0^u f^2(s)\,ds\right]^2 = 4\left(\frac{c}{g}\right)^2 \int_0^T dt \int_0^t du\,\text{arcsinh}^2\left(\frac{g}{c}u\right). \tag{19.16}$$

Now (19.13) and the substitution $((g/c)u = s)$ change this into

$$\begin{aligned}\sigma_\varepsilon^2 &= 4\left(\frac{c}{g}\right)^3 \int_0^T dt \Big[s\,\text{arcsinh}^2(s) - 2\sqrt{1+s^2}\,\text{arcsinh}(s) + 2s\Big]_0^{(\frac{g}{c}t)} \\ &= 4\left(\frac{c}{g}\right)^3 \left[\int_0^T \left(\frac{g}{c}t\right)\text{arcsinh}^2\left(\frac{g}{c}t\right)dt - 2\int_0^T \sqrt{1+\left(\frac{g}{c}t\right)^2}\,\text{arcsinh}\left(\frac{g}{c}t\right)dt + 2\int_0^T \frac{g}{c}t\,dt\right].\end{aligned}$$

The further substitution $(g/c)t = x$ and (19.14) yield

$$\begin{aligned}\sigma_\varepsilon^2 = 4\left(\frac{c}{g}\right)^4 &\left[\frac{1}{2}\left(\frac{g}{c}T\right)^2 \text{arcsinh}^2\left(\frac{g}{c}T\right) - \left(\frac{g}{c}T\right)\sqrt{1+\left(\frac{g}{c}T\right)^2}\,\text{arcsinh}\left(\frac{g}{c}T\right) + \frac{3}{2}\left(\frac{g}{c}T\right)^2\right. \\ &\left. - \int_0^{(\frac{g}{c}T)} \sqrt{1+x^2}\,\text{arcsinh}(x)\,dx\right]\end{aligned}$$

hence the integral (19.15) finally yields

$$\begin{aligned}\sigma_\varepsilon^2 = \left(\frac{c}{g}\right)^4 &\left[2\left(\frac{g}{c}T\right)^2 \text{arcsinh}^2\left(\frac{g}{c}T\right) - 4\left(\frac{g}{c}T\right)\sqrt{1+\left(\frac{g}{c}T\right)^2}\,\text{arcsinh}\left(\frac{g}{c}T\right)\right. \\ &\left. + 6\left(\frac{g}{c}T\right)^2 - 2\left(\frac{g}{c}T\right)\sqrt{1+\left(\frac{g}{c}T\right)^2}\,\text{arcsinh}\left(\frac{g}{c}T\right) + \left(\frac{g}{c}T\right)^2 - \text{arcsinh}\left(\frac{g}{c}T\right)\right].\end{aligned}$$

Rearranging, the total energy variance for the hyperbolic motion is obtained

$$\begin{aligned}\sigma_\varepsilon^2 = \left(\frac{c}{g}\right)^4 &\left\{\left[2\left(\frac{g}{c}T\right)^2 - 1\right]\text{arcsinh}^2\left(\frac{g}{c}T\right)\right. \\ &\left. - 6\left(\frac{g}{c}T\right)\sqrt{1+\left(\frac{g}{c}T\right)^2}\,\text{arcsinh}\left(\frac{g}{c}T\right) + 7\left(\frac{g}{c}T\right)^2\right\}.\end{aligned} \tag{19.17}$$

19.4 KLT FOR SIGNALS EMITTED IN ASYMPTOTIC HYPERBOLIC MOTION

The obvious asymptotic formula

$$\lim_{x\to\infty} \sqrt{1+x^2} = \lim_{x\to\infty} x$$

and its consequence

$$\lim_{x\to\infty} \ln[x + \sqrt{1+x^2}] = \lim_{x\to\infty} \ln[2x] \tag{19.18}$$

form the starting point to investigate the asymptotic hyperbolic motion. In fact, from (19.10), we see that, when $t \to \infty$, $X(t)$ approaches

$$B\left(\frac{c}{g}\ln\left(2\frac{g}{c}t\right)\right). \tag{19.19}$$

This we shall call the asymptotic hyperbolic motion and shall study it thoroughly.

By comparing (19.19) against (21.40), we immediately find

$$\int_0^t f^2(s)\,ds = \frac{c}{g}\ln\left(2\frac{g}{c}t\right). \tag{19.20}$$

Then, differentiating and taking the square root, we are led to

$$f(t) = \sqrt{\frac{c}{g}}\frac{1}{\sqrt{t}}. \tag{19.21}$$

This is the $f(t)$ function for the hyperbolic motion.

Integrating (19.21), one then gets

$$\int_0^t f(s)\,ds = 2\sqrt{\frac{c}{g}}\sqrt{t}. \tag{19.22}$$

By virtue of (19.21) and (19.22) the $\chi(t)$ function defined by (11.10) reads

$$\chi(t) = \sqrt{f(t)\int_0^t f(s)\,ds} = \sqrt{2\frac{c}{g}} \tag{19.23}$$

a constant. This circumstance is vital in order to develop the asymptotic hyperbolic case, inasmuch as it simplifies things greatly. In fact, from

$$\chi'(t) = 0 \tag{19.24}$$

and from (11.9), it can be seen at once that $\nu(t)$ vanishes identically

$$\nu(t) = 0 \tag{19.25}$$

(i.e., the order of the Bessel functions is zero). Thus, the KL expansion is given by functions of the form

$$J_0\left(\gamma_n \frac{\int_0^t f(s)\,ds}{\int_0^T f(s)\,ds}\right) = J_0\left(\gamma_n \frac{\sqrt{t}}{\sqrt{T}}\right). \tag{19.26}$$

Our next task is to find the meaning of the constants γ_n, formally given as the real positive zeros of (11.11). Letting $\chi'(t) = 0$ and $\nu'(t) = 0$, and getting rid of all multiplicative factors, one easily sees that (11.11) simplifies to

$$J_0'(\gamma_n) = 0. \tag{19.27}$$

Thus, the γ_n are the positive zeros, arranged in ascending order of magnitude, of the derivative of $J_0(x)$. In other words, they are the abscissas of the maxima and minima of $J_0(x)$, which are known to follow each other alternately. However, a different interpretation of the γ_n follows from the Bessel function property (see [7, p. 12, entry (55) (set $\nu = 0$)]

$$J'_\nu(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x). \tag{19.28}$$

In fact, (19.27) now becomes equivalent to

$$J_1(\gamma_n) = 0 \tag{19.29}$$

and one may also say that the γ_n are the real positive zeros of $J_1(x)$. The first 40 among them are listed in [8, p. 748], and one finds, for instance,

$$\gamma_1 = 3.8317060 \quad \gamma_2 = 7.0155867 \quad \gamma_{40} = 126.4461387. \tag{19.30}$$

No explicit formula yielding these zeros exactly is known. However, it is possible to get an approximated expression by setting $\nu = 1$ into the asymptotic formula for $J_\nu(x)$ (see [9, p. 134])

$$\lim_{x\to\infty} J_\nu(x) = \lim_{x\to\infty} \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{19.31}$$

from which

$$\cos\left(\gamma_n - \frac{3\pi}{4}\right) \approx 0 \tag{19.32}$$

or

$$\gamma_n - \frac{3\pi}{4} \approx n\pi - \frac{\pi}{2} \quad (n = 1, 2, \ldots). \tag{19.33}$$

Thus,

$$\gamma_n \approx n\pi + \frac{\pi}{4}. \tag{19.34}$$

We may see how good this approximation is by setting $n = 1, 2, \ldots, 40$

$$\gamma_1 \approx 3.9269908 \quad \gamma_2 \approx 7.0685835 \quad \gamma_{40} \approx 126.4491 \tag{19.35}$$

and checking these results against (19.30). Of course, the agreement improves with increasing n. As for the eigenvalues λ_n, they are related to the γ_n by (11.13)

$$\lambda_n = \frac{4cT}{g} \frac{1}{(\gamma_n)^2} \tag{19.36}$$

and are also variances of the independent Gaussian random variables Z_n.

Finally, we turn to the normalization constants N_n that are obtained from (11.12) after inserting (19.22) and (19.25). The resulting condition for N_n is

$$1 = N_n^2 \frac{4cT}{g} \int_0^1 x[J_0(\gamma_n x)]^2 \, dx. \tag{19.37}$$

This integral of (19.37) is calculated within the framework of the Dini expansion in series of Bessel functions (see [7, p. 71]), and one finds

$$\begin{aligned} 1 &= N_n^2 \frac{4cT}{g} \left\{ \frac{1}{2} \left[J_0'^2(\gamma_n) + \left(1 - \frac{0}{\gamma_n^2}\right) J_0^2(\gamma_n) \right] \right\} \\ &= N_n^2 \frac{2cT}{g} [J_0'^2(\gamma_n) + J_0^2(\gamma_n)] = N_n^2 \frac{2cT}{g} J_0^2(\gamma_n) \end{aligned} \tag{19.38}$$

where (19.27) was used in the last step. Solving with respect to N_n requires the introduction of the modulus of $J_0(\gamma_n)$, and one has

$$N_n = \frac{\sqrt{g}}{\sqrt{2cT}|J_0(\gamma_n)|}. \tag{19.39}$$

This is the exact expression of the normalization constants.

For an approximated expression for N_n, we substitute the Bessel function in its asymptotic form (19.31) with γ_n given in (19.34):

$$|J_0(\gamma_n)| \approx \left| \sqrt{\frac{2}{\pi\gamma_n}} \cos\left(\gamma_n - \frac{\pi}{4}\right) \right| = \sqrt{\frac{2}{\pi\gamma_n}} |\cos(n\pi)| = \sqrt{\frac{2}{\pi\gamma_n}}. \tag{19.40}$$

Then, from (19.39) and (19.40) we get the approximated N_n:

$$N_n \approx \frac{\pi}{2} \sqrt{\frac{g}{cT}} \sqrt{n + \tfrac{1}{4}}. \tag{19.41}$$

All the results obtained in this section may now be summarized by writing the exact KL expansion

$$B\left(\frac{c}{g} \ln\left(2\frac{g}{c}t\right)\right) = \sum_{n=1}^{\infty} Z_n \frac{\sqrt{c}}{\sqrt{g}} \cdot \frac{1}{\sqrt{2}\sqrt{T}|J_0(\gamma_n)|} J_0\left(\gamma_n \frac{\sqrt{t}}{\sqrt{T}}\right) \tag{19.42}$$

and the approximated expansion—found by virtue of (19.31) and (19.41)

$$B\left(\frac{c}{g} \ln\left(2\frac{g}{c}t\right)\right) = \sum_{n=1}^{\infty} Z_n \frac{\sqrt{c}}{\sqrt{g}} \cdot \frac{1}{\sqrt{2}T^{\frac{1}{4}}t^{\frac{1}{4}}} \cos\left(\gamma_n \frac{\sqrt{t}}{\sqrt{T}} - \frac{\pi}{4}\right). \tag{19.43}$$

The physical range of validity of (19.42) and (19.43) is provided by the relativistic condition (11.7). Since, from (19.21)

$$f^4(t) = \frac{c^2}{g^2} \cdot \frac{1}{t^2}, \tag{19.44}$$

(11.7) yields the velocity of the asymptotic hyperbolic motion

$$v(t) = c\sqrt{1 - \frac{c^2}{g^2} \cdot \frac{1}{t^2}}. \tag{19.45}$$

In order to have a non-negative radicand, the inequality

$$\frac{c^2}{g^2} \cdot \frac{1}{t^2} \leq 1 \tag{19.46}$$

must hold, meaning

$$t \geq \frac{c}{g} = 3.0612245 \cdot 10^7 \text{ s} \approx 0.96996974 \text{ years} \approx 1 \text{ year}. \tag{19.47}$$

Thus, the asymptotic approximation to the hyperbolic motion holds only after about 1 year of travel. Since any trip to even the nearest stars will certainly last longer than that, this approximation may be reagarded as physically acceptable.

19.5 CHECKING THE KLT OF ASYMPTOTIC HYPERBOLIC MOTION BY MATLAB SIMULATIONS

Just look at Figure 19.1.

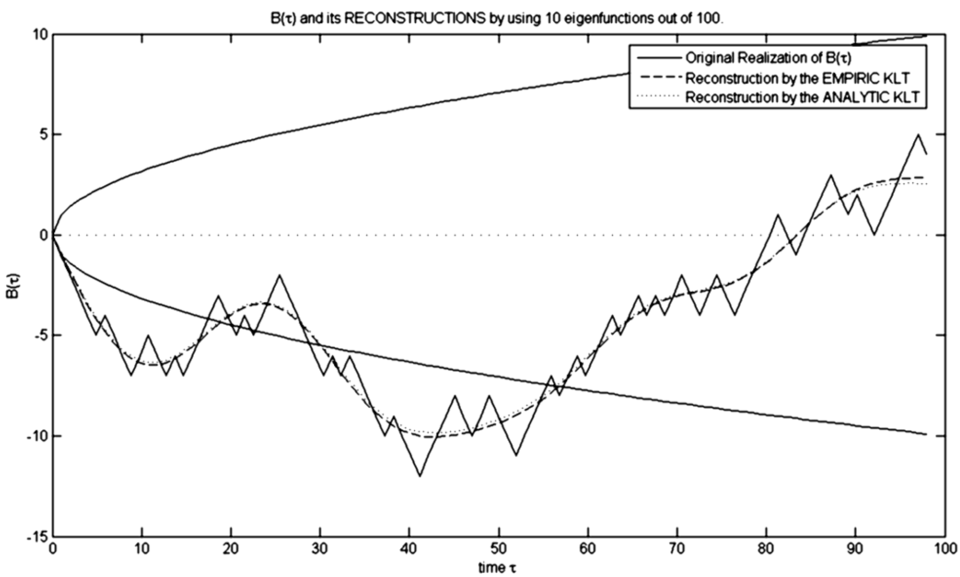

Figure 19.1. The time-rescaled Brownian motion $X(t)$ of (19.43) vs. time t simulated as a random walk over 100 time instants. This $X(t)$ represents the "noisy signal" received on Earth (whence the use of the coordinate time $t =$ Earth time) from a relativistic spaceship moving away from the Earth in an asymptotic hyperbolic motion, as in the science fiction novel *Tau Zero*. Next to the "bumpy curve" of $X(t)$, two more "smooth curves" are shown that *interpolate at best* the bumpy $X(t)$. These two curves are the KLT reconstruction of $X(t)$ *by using the first ten eigenfunctions only*. It is important to note that the two smooth curves are *different* in this case because the KLT expansion (19.43) is *approximated*. Actually, it is an approximated KLT expansion because the asymptotic expansion of the Bessel functions (19.31) was used. So, the two curves are different from each other, but both still interpolate $X(t)$ at best. Note that, were we taking into account the full set of 100 KLT eigenfuctions—rather than just 10—then the *empirical* reconstruction would overlap $X(t)$ *exactly*, but the *analytic* reconstruction would not because of the use of the asymptotic expansion (19.31) of the Bessel functions.

19.6 SIGNAL TOTAL ENERGY AS A STOCHASTIC PROCESS OF *T*

Formulas (21.60) and (19.20) enable us to obtain the total energy mean value

$$E\{\varepsilon_{Asy}\} = \int_0^T dt \int_0^t f^2(s)\, ds = \frac{c}{g}\int_0^T \ln\left(\frac{2g}{c}t\right) dt. \tag{19.48}$$

The substitution $x = (2g/c)t$ then results in

$$E\{\varepsilon_{Asy}\} = \frac{1}{2}\left(\frac{c}{g}\right)^2 [x(\ln x - 1)]_0^{(\frac{2g}{c})T} = \frac{cT}{g}\left[\ln\left(\frac{2g}{c}T\right) - 1\right].$$

Thus, the asymptotic mean total energy reads

$$E\{\varepsilon_{Asy}\} = \frac{cT}{g}\left[\ln\left(\frac{2g}{c}T\right) - 1\right]. \tag{19.49}$$

Note that the same asymptotic result is obtained from the exact expression (19.12) upon substituting arcsinh by log, and disregarding all the $+1$ that disappear for large T.

Next let us turn to the asymptotic total energy variance by resorting to

$$\int \ln^2 x\, dx = x\ln^2 x - 2x\ln x + 2x + C \tag{19.50}$$

$$\int x\ln^2 x\, dx = \frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x + \frac{x^2}{4} + C \tag{19.51}$$

$$\int x\ln x\, dx = \frac{x^2}{2}\ln x - \frac{x^2}{4} + C. \tag{19.52}$$

In fact, inserting (19.20) into the expression for σ_ε^2 in (21.62), one finds

$$\begin{aligned} \sigma^2_{\varepsilon_{Asy}} &= 4\int_0^T dt \int_0^t du \left[\int_0^u f^2(s)\, ds\right]^2 \\ &= 4\left(\frac{c}{g}\right)^2 \int_0^T dt \int_0^t \ln^2\left(\frac{2g}{c}u\right) du \end{aligned} \tag{19.53}$$

whence the substitution $[(2g)/c]u = x$ and the integral in (19.50) yield

$$\sigma^2_{\varepsilon_{Asy}} = 2\left(\frac{c}{g}\right)^3 \int_0^T dt[x\ln^2 x - 2x\ln x + 2x]_0^{\frac{2g}{c}t}.$$

The further substitution $[(2g/c)]t = x$ now leads to the couple of integrals (19.51) and (19.52)

$$\sigma^2_{\varepsilon_{Asy}} = \left(\frac{c}{g}\right)^4 \left[\int_0^{\frac{2g}{c}T} x \ln^2 x \, dx - 2\int_0^{\frac{2g}{c}T} x \ln x \, dx + 2\int_0^{\frac{2g}{c}T} x \, dx\right]$$

$$= \left(\frac{c}{g}\right)^4 \left[\frac{x^2}{2}\ln^2 x - \frac{x^2}{2}\ln x - x^2 \ln x + \frac{x^2}{4} + \frac{x^2}{2} + x^2\right]_0^{\frac{2g}{c}T}$$

$$= \left(\frac{c}{g}\right)^4 \frac{1}{4}\left[x^2(2\ln^2 x - 6\ln x + 7)\right]_0^{\frac{2g}{c}T}$$

$$= \left(\frac{cT}{g}\right)^2 \left[2\ln^2\left(\frac{2g}{c}T\right) - 6\ln\left(\frac{2g}{c}T\right) + 7\right].$$

Thus, the asymptotic total energy variance reads

$$\sigma^2_{\varepsilon_{Asy}} = \left(\frac{cT}{g}\right)^2 \left[2\ln^2\left(\frac{2g}{c}T\right) - 6\ln\left(\frac{2g}{c}T\right) + 7\right]. \tag{19.54}$$

Note that just as (19.49) is the asymptotic version of (19.12), so (19.54) is the asymptotic form of (19.17), and could have been found by substituting arcsinh by log, and forgetting all the additive $+1$ that are dwarfed for large T.

The square root of (19.74) is the asymptotic total energy standard deviation

$$\sigma_{\varepsilon_{Asy}} = \pm\frac{cT}{g}\sqrt{2\ln^2\left(\frac{2g}{c}T\right) - 6\ln\left(\frac{2g}{c}T\right) + 7}. \tag{19.55}$$

Setting

$$\frac{gT}{c} = x \tag{19.56}$$

we see that the radicand of (19.55) is the quadratic in $\xi \equiv \ln x$

$$2\xi^2 - 6\xi + 7 > 0. \tag{19.57}$$

This is positive for any ξ because $\Delta = -20 < 0$.

Let us regard the noise asymptotic total energy as a stochastic process of T. The process behavior in time is characterized by its mean value curve (19.49) and by the upper and lower (mean value $\pm$ standard deviation) curves given by

$$E\{\varepsilon_{Asy}\} \pm \sigma_{\varepsilon_{Asy}}. \tag{19.58}$$

The first column of Table 19.1 shows the numerical values of the independent variable x defined by (19.56) ranging from 0 to 20. In units of time, T ranges from 0 to 20 years since

$$\frac{c}{g} \approx 3.0612245 \cdot 10^7 \text{ s} \approx 0.96699947 \text{ years} \approx 1 \text{ year}. \tag{19.59a}$$

Table 19.1. Noise asymptotic total energy.

$$x = \frac{gT}{c}$$

$$M = x(\ln(2x) - 1)$$

$$V = x^2(2\ln^2(2x) - 6\ln(2x) + 7)$$

x	M	$M - \sqrt{V}$	$M + \sqrt{V}$
0	0	0	0
1	−0.30685	−2.25673	1.643024
2	0.772588	−2.40600	3.951178
3	2.375278	−2.52698	7.277545
4	4.317766	−2.80572	11.44125
5	6.512925	−3.21883	16.24468
6	8.909439	−3.73346	21.55234
7	11.47340	−4.32692	27.27327
8	14.18070	−4.98419	33.34561
9	17.01334	−5.69497	39.72166
10	19.95732	−6.45182	46.36646
11	23.00146	−7.24919	53.25213
12	26.13664	−8.08277	60.35607
13	29.35525	−8.94912	67.65963
14	32.65086	−9.84541	75.14714
15	36.01796	−18.7693	82.80522
16	39.45177	−11.7188	90.62235
17	42.94812	−12.6922	98.58846
18	46.50334	−13.6880	106.6947
19	50.11413	−14.7049	114.9332
20	53.77758	−15.7418	123.2970

The second column gives the numerical values of the asymptotic mean value (19.49) of the noise total energy apart from a factor $(c/g)^2$. The third and fourth columns, respectively, show the values of the lower (minus sign) and upper (plus sign) curves (19.58), again apart from a factor $(c/g)^2$.

One may check the above asymptotic total energy results against the corresponding exact results derived at the end of Section 19.3. Table 19.2 shows the same items as Table 19.1, but is calculated by using the exact total energy variance (19.17). We see that the agreement is not as good for very small values of T, while it increases for increasing T, and the dispersion of the total energy around its mean value increases roughly by the same amount as the total energy itself.

The conclusion to this section is that the KL eigenfunction expansion has been derived for the noise emitted by a spaceship traveling at a constantly accelerated relativistic motion. Though the mathematical difficulties forced us to confine ourselves to the asymptotic theory for values of time larger than 1 year, the study of the noise total energy (where both asymptotic and exact results can be obtained) shows that the errors of the asymptotic version are not very large.

19.7 INSTANTANEOUS NOISE ENERGY FOR ASYMPTOTIC HYPERBOLIC MOTION: PREPARATORY CALCULATIONS

In Chapter 24, as well as in [2], the process $Y(t)$ defined by

$$Y(t) = X^2(t) - E\{X^2(t)\} \tag{19.59b}$$

was considered. According to (24.35), the KL eigenfunction expansion of that process reads

$$Y(t) = \sum_{n=1}^{\infty} \tilde{Z}_n \tilde{N}_n \sqrt{\tilde{f}(t) \int_0^t \tilde{f}(s)\,ds} \cdot J_{\tilde{\nu}(t)}\left(\tilde{\gamma}_n \frac{\displaystyle\int_0^t \tilde{f}(s)\,ds}{\displaystyle\int_0^T \tilde{f}(s)\,ds}\right), \tag{19.60}$$

where the function $\tilde{f}(t)$ is defined in terms of $f(t)$ by (24.24). That is,

$$\tilde{f}(t) \equiv 2f(t)\sqrt{\int_0^t f^2(z)\,dz}. \tag{19.61}$$

This section is devoted to finding the KL expansion of the zero-mean square process $Y(t)$, in the asymptotic hyperbolic case, and its physical meaning for relativistic interstellar flight will be examined in the coming section. In this section we just pave the mathematical way to the coming section by performing the necessary calculations.

Table 19.2. Noise exact total energy.

$$x = \frac{gT}{c}$$

$$M = x\ln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2} + 1$$

$$V = (2x^2 - 1)[\ln^2(x + \sqrt{1 + x^2}) - 6x\sqrt{1 + x^2}\ln(x + \sqrt{1 + x^2}) + 7x^2]$$

0	0	0	0
1	0.467160	−0.07884	1.013160
2	1.651202	−0.31139	3.613797
3	3.293061	−0.67013	7.256257
4	5.255744	−1.12713	11.63862
5	7.463172	−1.66295	16.58929
6	9.867916	−2.26419	22.00002
7	12.43777	−2.92124	27.79679
8	15.14952	−3.62692	33.92596
9	17.98561	−4.37568	40.34690
10	20.93235	−5.16310	47.02780
11	23.97876	−5.98558	53.94311
12	27.11583	−6.84016	61.07182
13	30.33603	−7.72432	68.39640
14	33.633301	−8.63592	75.90195
15	37.00130	−9.57310	83.57571
16	40.43615	−18.5342	91.40657
17	43.93342	−11.5179	99.38482
18	47.48945	−12.5229	107.5018
19	51.10098	−13.5481	115.7500

According to (19.61), we must first obtain the function $\tilde{f}(t)$, which follows at once from (19.20) and (19.21)

$$\tilde{f}(t) = \frac{2c}{g}\sqrt{\frac{\ln\left(\frac{g}{c}t\right)}{t}}. \tag{19.62}$$

We now proceed to construct the complicated expression (4.26), or, alternatively, (3.50) with $f(t)$ substituted by $\tilde{f}(t)$, to find the time-dependent order $\tilde{\nu}(t)$. But a glance at (24.26) and (19.62) shows that considerable analytical difficulties are involved. For instance, evaluation of the integral appearing in (22.50) with $f(t)$ substituted by $\tilde{f}(t)$, namely

$$\int_0^t \tilde{f}(s)\,ds = \int_0^t \frac{2c}{g}\sqrt{\frac{\ln\left(2\frac{g}{c}s\right)}{s}}\,ds \tag{19.63}$$

does not seem to be feasible in terms of elementary transcendental functions.

Nevertheless, these difficulties may be overcome by keeping in mind that we are seeking the asymptotic version of (19.62) for large values of time. Therefore, one is led to consider the limit

$$\lim_{t\to\infty}\tilde{f}(t) = \lim_{t\to\infty}\frac{2c}{g}\sqrt{\frac{\ln\left(2\frac{g}{c}t\right)}{t}} = \frac{2c}{g}\sqrt{\lim_{t\to\infty}\frac{\ln\left(2\frac{g}{c}t\right)}{t}} = \frac{\infty}{\infty} \tag{19.64}$$

where the indefinite form forces us to resort to L'Hospital's rule, and yields

$$\lim_{t\to\infty}\tilde{f}(t) = \frac{2c}{g}\sqrt{\lim_{t\to\infty}\frac{\left(2\frac{g}{c}\right)(1)}{\left(2\frac{g}{c}t\right)(1)}} = \frac{2c}{g}\lim_{t\to\infty}\frac{1}{\sqrt{t}} = \lim_{t\to\infty}\frac{2c}{g}\frac{1}{\sqrt{t}}.$$

Concluding the calculation at the last limit, and checking this against the initial limit, we thus obtain the following "ultimate" asymptotic version of (19.62), which from now on we shall regard as the asyptotic replacement to (19.62) for large values of time t

$$\tilde{f}(t) = \frac{2c}{g}\frac{1}{\sqrt{t}}. \tag{19.65}$$

This formula is simple enough for us to perform the remaining calculations involved with the KL expansion (19.60).

Beginning with the computation of the Bessel function order (22.50) with $f(t)$ substituted by $\tilde{f}(t)$, (19.65) yields at once

$$\ln\tilde{f}(t) = \ln\left(\frac{2c}{g}\right) - \tfrac{1}{2}\ln t \tag{19.66}$$

whence

$$\frac{d \ln \tilde{f}(t)}{dt} = -\frac{1}{2}\frac{1}{t} \tag{19.67}$$

$$\frac{d^2 \ln \tilde{f}(t)}{dt^2} = +\frac{1}{2}\frac{1}{t^2} \tag{19.68}$$

and

$$\int_0^t \tilde{f}(s)\, ds = \frac{2c}{g}\int_0^t \frac{1}{\sqrt{s}} ds = \frac{4c}{g}\sqrt{t}. \tag{19.69}$$

Therefore, (22.50), with $f(t)$ substituted by $\tilde{f}(t)$, yields

$$\begin{aligned}\tilde{\nu}(t) &= \sqrt{\frac{1}{4} + \left[\frac{\frac{4c}{g}\sqrt{t}}{\frac{2c}{g}\frac{1}{\sqrt{t}}}\right]^2 \left\{\frac{3}{4}\left(-\frac{1}{2}\frac{1}{t}\right)^2 - \frac{1}{2}\left(\frac{1}{2}\frac{1}{t^2}\right)\right\}} \\ &= \sqrt{\frac{1}{4} + [2t]^2\left\{\frac{3}{16}\frac{1}{t^2} - \frac{1}{4}\frac{1}{t^2}\right\}} = \sqrt{\frac{1}{4} + 4t^2 \cdot \frac{1}{t^2}\left\{\frac{3}{16} - \frac{1}{4}\right\}}.\end{aligned} \tag{19.70}$$

The time variable is thus seen to disappear from the last formula, leaving

$$\tilde{\nu}(t) = \sqrt{\frac{1}{4} + 4\left\{\frac{3-4}{16}\right\}} = \sqrt{\frac{1}{4} + \frac{-1}{4}} = \sqrt{0} = 0.$$

That is, the order of the Bessel function vanishes identically

$$\tilde{\nu}(t) = 0 \tag{19.71}$$

and this circumstance helps to simplify further calculations considerably. Intuitively speaking, (19.71) is quite a reasonable result. In fact, on the one hand, the corresponding Bessel function order in the KL expansion of the $X(t)$ process vanished too

$$\nu(t) = 0, \tag{19.72}$$

which is Equation (19.25), or eq. (68) in [1]. On the other hand, (19.71) truly mirrors the asymptotic character of the KL expansion under consideration, since the Bessel function of order zero is the only Bessel function of the first kind to have its initial value equal to one rather than zero, pointing out the non-validity of this theory for values of time near to the origin.

Let us now proceed to finding the function $\tilde{\chi}(t)$ defined by (24.25). By virtue of (19.65) and (19.69), it follows that

$$\tilde{\chi}(t) = \sqrt{\frac{2c}{g}\frac{1}{\sqrt{t}} \cdot \frac{4c}{g}\sqrt{t}} = 2\sqrt{2}\frac{c}{g}. \tag{19.73}$$

Once again, the time variable cancels out from the last formula, yielding a constant rather than a time function. An immediate consequence of (19.73) is, of course,

$$\tilde{\chi}'(t) = 0 \tag{19.74}$$

which helps to simplify further calculations also.

Reverting now to the KL expansion of (19.60), we see that the Bessel function must have the form

$$J_0\left(\tilde{\gamma}_n \frac{\displaystyle\int_0^t \tilde{f}(s)\,ds}{\displaystyle\int_0^T \tilde{f}(s)\,ds}\right) = J_0\left(\tilde{\gamma}_n \frac{\sqrt{t}}{\sqrt{T}}\right). \tag{19.75}$$

Our next task is to find the meaning of the constants $\tilde{\gamma}_n$, given by (24.27). As $\tilde{\chi}'(t) = 0$ and $\tilde{\nu}'(t) = 0$, and getting rid of all multiplicative factors, one easily sees that (24.27) yields

$$J_0'(\tilde{\gamma}_n) = 0. \tag{19.76}$$

Thus, the $\tilde{\gamma}_n$ are the positive zeros, arranged in ascending order of magnitude, of the derivative of $J_0(x)$. In other words, they are the abscissas of the maxima and minima of $J_0(x)$, that are known to follow each other alternately. However, a different interpretation of the $\tilde{\gamma}_n$ follows from (see [7, p. 12, entry 55 ($\nu = 0$ must be set)]

$$J_\nu'(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x). \tag{19.77}$$

Using (19.77), (19.76) now becomes equivalent to

$$J_1(\tilde{\gamma}_n) = 0 \tag{19.78}$$

and one may also say that the $\tilde{\gamma}_n$ are the real positive zeros of $J_1(x)$ The first 40 among them are listed in [8, p. 748], and one finds, for instance:

$$\tilde{\gamma}_1 = 3.8317060 \quad \tilde{\gamma}_2 = 7.0155867 \quad \tilde{\gamma}_{40} = 126.4461387. \tag{19.79}$$

No explicit formula yielding these zeros exactly is known. However, it is possible to get an approximated expression for them on setting $\nu = 1$ into the asymptotic formula for $J_\nu(x)$ (see [9, p. 134])

$$\lim_{x\to\infty} J_\nu(x) = \lim_{x\to\infty} \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \tag{19.80}$$

getting

$$\cos\left(\tilde{\gamma}_n - \frac{3\pi}{4}\right) \approx 0 \tag{19.81}$$

whence

$$\tilde{\gamma}_n - \frac{3\pi}{4} \approx n\pi - \frac{\pi}{2} \quad (n = 1, 2, \ldots) \tag{19.82}$$

and finally

$$\tilde{\gamma}_n \approx n\pi + \frac{\pi}{4}. \tag{19.83}$$

We can see how good this approximation is by setting $n = 1, 2, \ldots, 40$,

$$\tilde{\gamma}_1 \approx 3.9269908 \quad \tilde{\gamma}_2 \approx 7.0685835 \quad \tilde{\gamma}_{40} \approx 126.4491 \tag{19.84}$$

and checking these results against (19.79): the agreement improves with increasing n.

As for the eigenvalues $\tilde{\lambda}_n$, they are related to the $\tilde{\gamma}_n$ by (24.29), and, by virtue of (19.69), take the form

$$\tilde{\lambda}_n = \frac{16c^2 T}{g^2} \frac{1}{(\tilde{\gamma}_n)^2}. \tag{19.85}$$

Finally, we turn to the normalization constants $\tilde{N}_n$ that can be discovered from (24.28) by inserting (19.69) and (19.71). Therefore

$$1 = \tilde{N}_n^2 \frac{16c^2 T}{g^2} \int_0^1 x[J_0(\tilde{\gamma}_n x)]^2 \, dx. \tag{19.86}$$

This integral is evaluated within the framework of the Dini expansion in the series of Bessel functions [5, p. 71], and one finds

$$\begin{aligned} 1 &= \tilde{N}_n^2 \frac{16c^2 T}{g^2} \left\{ \frac{1}{2} \left[J_0'^2(\tilde{\gamma}_n) + \left(1 - \frac{0}{\tilde{\gamma}_n^2}\right) J_0^2(\tilde{\gamma}_n) \right] \right\} \\ &= \tilde{N}_n^2 \frac{8c^2 T}{g^2} [J_0'^2(\tilde{\gamma}_n) + J_0^2(\tilde{\gamma}_n)] = \tilde{N}_n^2 \frac{8c^2 T}{g^2} J_0^2(\tilde{\gamma}_n) \end{aligned} \tag{19.87}$$

where (19.76) was used in the last step. Solving for $\tilde{N}_n$ requires the introduction of the modulus of $J_0(\tilde{\gamma}_n)$, and one has

$$\tilde{N}_n = \frac{g}{2\sqrt{2}c\sqrt{T}|J_0(\tilde{\gamma}_n)|}. \tag{19.88}$$

This is the exact expression of the normalization constants. We can, however, derive an approximated expression for them on substituting the Bessel function by virtue of (19.80) and (19.83)

$$|J_0(\tilde{\gamma}_n)| \approx \left| \sqrt{\frac{2}{\pi \tilde{\gamma}_n}} \cos\left(\tilde{\gamma}_n - \frac{\pi}{4}\right) \right| = \sqrt{\frac{2}{\pi \tilde{\gamma}_n}} |\cos(n\pi)| = \sqrt{\frac{2}{\pi \tilde{\gamma}_n}}. \tag{19.89}$$

Thus, from (19.88), by virtue of (19.89) and (19.83), we get the approximated $\tilde{N}_n$:

$$\tilde{N}_n \approx \frac{\pi}{4} \frac{g}{c\sqrt{T}} \sqrt{n + \tfrac{1}{4}} \tag{19.90}$$

which completes our set of preliminary calculations.

19.8 KL EXPANSION FOR THE INSTANTANEOUS ENERGY OF THE NOISE EMITTED BY A RELATIVISTIC SPACESHIP

When dealing with a noise represented by a stochastic process $X(t)$, an important distinction is between its instantaneous energy, given by the square process

$$X^2(t) \tag{19.91}$$

and the total energy, given by the stochastic integral of the instantanous energy (19.91) over the finite time span, $0 \le t \le T$ during which the noise is observed:

$$I = \int_0^T X^2(s)\, ds. \tag{19.92}$$

This section is devoted to finding the KL expansion of the process (19.91), whereas both mean value and variance of the random variable (19.92) have already been obtained in Section 19.3, as well as in section 5 of [1]. A related paper, [10], may also be consulted.

Let us then consider the mean value of (19.91), given by (21.59); that is,

$$E\{X^2(t)\} = \int_0^t f^2(s)\, ds, \tag{19.93}$$

where E denotes mean value operator, or ensemble average. By virtue of (19.20), (19.93) is changed into

$$E\{X^2(t)\} = \frac{c}{g}\ln\left(2\frac{g}{c}t\right). \tag{19.94}$$

Thus, the zero-mean square process $Y(t)$, defined by (19.59), takes the form

$$Y(t) = X^2(t) - \frac{c}{g}\ln\left(2\frac{g}{c}t\right) \tag{19.95}$$

whence

$$X^2(t) = \frac{c}{g}\ln\left(2\frac{g}{c}t\right) + Y(t). \tag{19.96}$$

Let us now consider the KL expansion of the $Y(t)$ process. By sustituting into (19.60) the normalization constants (19.90), the $\tilde{\chi}(t)$ function (19.73), and the Bessel function (19.75), we come up with

$$Y(t) = \sum_{n=1}^{\infty} \tilde{Z}_n \frac{g}{2\sqrt{2}c\sqrt{T}|J_0(\tilde{\gamma}_n)|} 2\sqrt{2}\frac{c}{g} J_0\left(\tilde{\gamma}_n \frac{\sqrt{t}}{\sqrt{T}}\right)$$

from which both c and g disappear, yielding

$$Y(t) = \sum_{n=1}^{\infty} \tilde{Z}_n \frac{1}{\sqrt{T}|J_0(\tilde{\gamma}_n)|} J_0\left(\tilde{\gamma}_n \frac{\sqrt{t}}{\sqrt{T}}\right). \tag{19.97}$$

Thus, by virtue of (19.96) and (19.97), we conclude that the exact KL expansion of the instantaneous energy $X^2(t)$ reads

$$X^2(t) = \frac{c}{g}\ln\left(2\frac{g}{c}t\right) + \sum_{n=1}^{\infty} \tilde{Z}_n \frac{1}{\sqrt{T}|J_0(\tilde{\gamma}_n)|}\frac{2\sqrt{2}c}{g} J_0\left(\tilde{\gamma}_n \frac{\sqrt{t}}{\sqrt{T}}\right). \tag{19.98}$$

From this exact expansion we may also derive an approximated one by resorting to the usual asymptotic formula (19.80) for both the Bessel functions appearing in (19.98). The result is

$$X^2(t) \approx \frac{c}{g}\ln\left(2\frac{g}{c}t\right) + \sum_{n=1}^{\infty} \tilde{Z}_n \frac{1}{T^{\frac{1}{4}}t^{\frac{1}{4}}}\frac{2\sqrt{2}c}{g}\cos\left(\tilde{\gamma}_n \frac{\sqrt{t}}{\sqrt{T}} - \frac{\pi}{4}\right), \tag{19.99}$$

which, after substituting the $\tilde{\gamma}_n$ by the approximated version (19.83), takes the final form

$$X^2(t) \approx \frac{c}{g}\ln\left(2\frac{g}{c}t\right) + \sum_{n=1}^{\infty} \tilde{Z}_n \frac{1}{T^{\frac{1}{4}}t^{\frac{1}{4}}}\frac{2\sqrt{2}c}{g}\cos\left(\left(n\pi + \frac{\pi}{4}\right)\frac{\sqrt{t}}{\sqrt{T}} - \frac{\pi}{4}\right). \tag{19.100}$$

This is the approximated (i.e., asymptotic) KL expansion of the noise instantaneous energy for large values of time. The computational advantage of (19.100) over (19.98) is that the Bessel functions have been substituted by a cosine.

19.9 CONCLUSION

A surprising property of both the instantaneous energy KL expansions (19.98) and (19.100) is revealed by checking them, respectively, against the corresponding KL expansions (19.42) and (19.43) of the noise process $X(t)$. In fact, on the one hand, one should note that the $\tilde{\gamma}_n$ (19.78) of the $Y(t)$ process are just the same as the $\tilde{\gamma}_n$ of the $X(t)$ process, given by (19.29), inasmuch as both are the real positive zeros of $J_1(t)$. Moreover, a glance shows that (19.98) has just the same eigenfunctions as (19.42), and (19.100) as (19.43). Therefore, we reach the unexpected conclusion that, when dealing with the noise emitted by a relativistic spaceship in asymptotic hyperbolic motion, the best orthonormal basis in the Hilbert space (i.e., the basis spanned by the eigenfunctions) is the same for both the noise and its own zero-mean instantaneous energy. Alternatively, if we prefer to give up the zero-mean restriction, we may say that the noise and its own instantaneous energy share parallel optimal reference frames, or bases, in the Hilbert space. This unusual feature should bear consequences in the design of a correct signal analysis procedure to filter out the noise received on Earth from a relativistically moving spaceship in asymptotic hyperbolic motion.

19.10 REFERENCES

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