Estimating ASKAP beam to beam correlation

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Summary

The radiometer noise in an ASKAP electronically formed beam is, in general, not independent of the noise in other beams formed on the same antenna. Each beam is formed from a subset of the 188 receptors on each PAF so that beams that share receptors also have some common noise. The correlation in noise between beams has been measured (Heywood et al. 2016), and in this memorandum a simple beam model is used to provide a means of predicting the degree of correlation.

1 Introduction

ASKAP forms beams electronically as linear combinations of outputs of a set of PAF elements, with beam properties controlled by the choice of weights. There is a mapping from the position of PAF elements in the focal plane 'illuminated' by the weights to the angular position of the beam on the sky. Antenna optics and the element radiation patterns determine the mapping. In general, the sets of elements used for two overlapping beams will themselves overlap. Any elements that are used in both beams will contribute their receiver noise to both, so there will be non-zero correlation between the signals produced by the beamformers for the two beams. The number of shared elements, and therefore the degree of correlation between two beams is expected to increase with decreasing angular separation.

The consequence of this correlation is the non-independence of image noise in images from different beams, leading to reduced sensitivity in the final mosaic relative to that of a mosaic contructed from truly independent images. The magnitude of the effect has been measured and reported by Serra et al. (2015), with more detail given in Heywood et al. (2016).

The optimal design of beam footprints depends on a number of factors: beam size relative to PAF field-of-view, how to properly sample the sky brightness distibution, how to tile a sky survey with footprints, and the sensitivity implications of beam-to-beam correlation. The need for a quantitative estimate of beam-to-beam correlation for a given footprint motivates this note, in which I attempt to provide a means for making that estimate.

2 Visibility statistics

The statistical properties (including correlation) of the images are determined by those properties of the contributing visibility measurements. In this section I make some simplifying assumptions:

- that identical beams are formed on each antenna and the corresponding beam weights are also identical;
- that the PAF elements are ideal, having the same gain and the same mutually independent receiver noise.

We want to determine the covariance $\mathrm{Cov}[V_{Ajk}, V_{Bjk}]$ where V_{Ajk},V_{Bjk} are the visibilities measured between antennas j,k on beams A,B.

$$V_{Ajk} =< v_{Aj} \cdot v_{Ak} >, V_{Bjk} =< v_{Bj} \cdot v_{Bk} > \tag{1}$$

where v_{Aj} is the voltage from beamformer B on antenna j, and so on. Then

$$\begin{aligned}\mathrm{Cov}[V_{Ajk}, V_{Bjk}] &= \mathrm{E}[V_{Ajk} \cdot V_{Bjk}] \\ &= \mathrm{E}[v_{Aj} \cdot v_{Ak} \cdot v_{Bj} \cdot v_{Bk}] \\ &= \mathrm{E}[v_{Aj} \cdot v_{Bj} \cdot v_{Ak} \cdot v_{Bk}].\end{aligned} \tag{2}$$

The correlation between V_A and V_B arises from the non-independence of v_{Aj} and v_{Bj}, and of v_{Ak} and v_{Bk}.

The beamformer voltages are

$$v_A = \sum_i w_{Aj_i} x_i \qquad v_B = \sum_i w_{Bj_i} x_i \tag{3}$$

where the x_i is the output from PAF element i and w_{Aj_i} and w_{Bj_i} are the correpsonding beamformer weights for beams A and B on antenna j. The x_i are assumed to be random variables with $\mathrm{E}[x_i] = 0$, $\mathrm{Var}[x_i] = s_x$, and $\mathrm{Cov}[x_{i_1}, x_{i_2}] = 0$ for $i_1 \neq i_2$.

The variance and covariance of the beamformer voltages are (dropping the antenna subscript j):

$$\begin{aligned}
\mathrm{Var}[v_A] &= s_x \sum_i w_{A_i}^2 \\
\mathrm{Var}[v_B] &= s_x \sum_i w_{B_i}^2 \\
\mathrm{Cov}[v_A, v_B] &= s_x \sum_i w_{A_i} w_{B_i}
\end{aligned} \tag{4}$$

Using these, we can write the variance and covariances of the visibilities:

$$\begin{aligned}
\mathrm{Var}[V_A] &= s_x^2 (\sum_i w_{A_i}^2)^2 \quad \text{assuming weights are antenna-independent} \\
\mathrm{Var}[V_B] &= s_x^2 (\sum_i w_{B_i}^2)^2 \\
\mathrm{Cov}[V_A, V_B] &= s_x^2 (\sum_i w_{A_i} w_{B_i})^2
\end{aligned} \tag{5}$$

The correlation coefficient R_{AB} between visibilities from beams A and B is

$$R_{AB} = \frac{(\sum_i w_{A_i} w_{B_i})^2}{\sum_i w_{A_i}^2 \sum_i w_{B_i}^2} \tag{6}$$

2.1 The F ratio

Heywood (Heywood et al. 2016) defines the ratio F to be $F = \frac{\sigma_c}{\sigma_i}$, where σ_c is the image formed by combining signals from correlated beams, and σ_i is the noise of images formed from that same beams, but made independent by sampling at different times. Then

$$F = \sqrt{1 + R_{AB}} \tag{7}$$

2.2 Modelling the beam weights

To use the correlation expression 6 we need beam weights determined on the telescope, or a means to model them for some proposed beam footprint. Here I attempt to construct a simple model. To date, all ASKAP beams have been formed using the maximum sensitivity (maxSNR) method, which accounts for both the pattern of received signal across the PAF and for

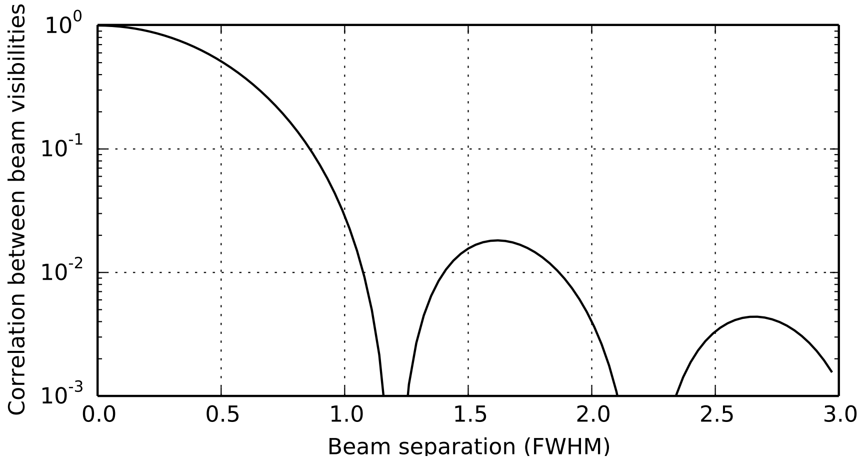

Figure 1: The correlation coefficient coefficient between visibilities for two beams as a function of their separation. Beam weights for each PAF element are assumed to be proportional to the amplitude of the signal received by that element.

the noise environment (including element-to-element noise variations and additive spillover noise) Jeffs et al. (2008). For the present purposes, I suppose that representative weights could be determined by Conjugate Field Match (CFM) (Jeffs et al.), which can be estimated from knowledge of the field distribution in the focal plane.

For CFM weights, an element is weighted according to the amplitude of its received signal, which I model as the voltage corresponding to an Airy pattern. Fig. 1 shows the correlation coefficient R_{AB} from expression 6, as a function of beam separation.

Fig. 2 uses this model to predict values of the F ratio measured and reported by Heywood et al. (2016). It can be directly compared with Fig. 3 of that memo.

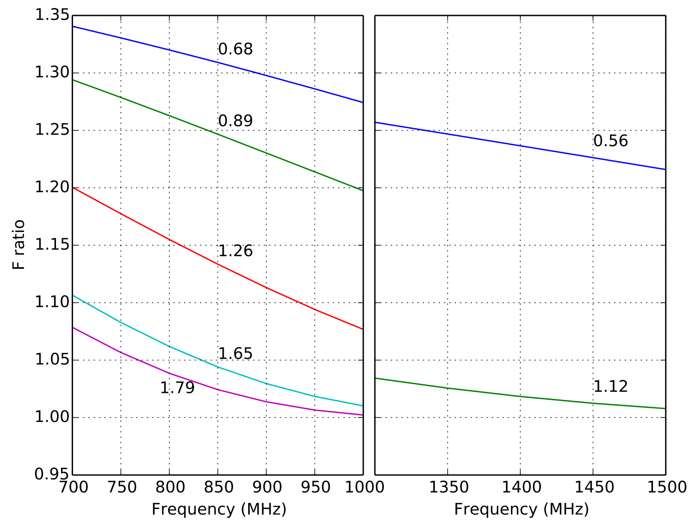

Figure 2: The F ratio for the cases measured by Heywood (left) and Serra (right). Each line is labelled with the beam separation in degrees. This figure can be compared directly with Fig. 3 of Heywood et al. (2016)

A Derivation of some expressions

The variance and covariance of the beamformer voltages are:

$$\begin{aligned}
\mathrm{Var}[v_A] &= \mathrm{Var}[\sum_i w_{A_i} x_i] \\
&= \sum_i \mathrm{Var}[w_{A_i} x_i] \\
&= \sum_i w_{A_i}^2 \,\mathrm{Var}[x_i] \\
&= s_x \sum_i w_{A_i}^2 \\
\mathrm{Var}[v_B] &= s_x \sum_i w_{B_i}^2
\end{aligned} \tag{8}$$

and

$$\begin{aligned}
\mathrm{Cov}[v_A, v_B] &= \mathrm{Cov}[\sum_i w_{A_i} x_i, \sum_i w_{B_i} x_i] \\
&= \mathrm{Cov}[(w_{A0}x_0 + w_{A1}x_1 + \cdots), (w_{B0}x_0 + w_{B1}x_1 + \cdots)] \\
&= \mathrm{Cov}[w_{A0}x_0, w_{B0}x_0] + \mathrm{Cov}[w_{A1}x_1, w_{B1}x_1] + \cdots \\
&= w_{A0}w_{B0}\,\mathrm{Var}[x_0] + w_{A1}w_{B1}\,\mathrm{Var}[x_1] + \cdots \\
&= s_x \sum_i w_{A_i} w_{B_i}
\end{aligned} \tag{9}$$

$$\begin{aligned}
\mathrm{Var}[V_A] &= \mathrm{Var}[v_{Aj} \cdot v_{Ak}] \\
&= \mathrm{E}[v_{Aj}^2 \cdot v_{Ak}^2] - \mathrm{Cov}[v_{Aj}, v_{Ak}] \\
&= s_x^2 \sum_i w_{Aj_i}^2 \sum_i w_{Ak_i}^2 \\
&= s_x^2 (\sum_i w_{A_i}^2)^2 \quad \text{assuming weights are antenna-independent} \\
\mathrm{Var}[V_B] &= s_x^2 (\sum_i w_{B_i}^2)^2
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\mathrm{Cov}[V_A, V_B] &= \mathrm{E}[v_{Aj} \cdot v_{Bj} \cdot v_{Ak} \cdot v_{Bk}] \\
&= \mathrm{E}[v_{Aj} \cdot v_{Bj}] \cdot \mathrm{E}[v_{Ak} \cdot v_{Bk}] + \mathrm{Cov}[v_{Aj} \cdot v_{Bj}, v_{Ak} \cdot v_{Bk}] \\
&= [\mathrm{E}[v_{Aj}] \cdot \mathrm{E}[v_{Bj}] + \mathrm{Cov}[v_{Aj}, v_{Bj}]] \cdot [\mathrm{E}[v_{Ak}] \cdot \mathrm{E}[v_{Bk}] + \mathrm{Cov}[v_{Ak}, v_{Bk}]] \\
&= s_x \sum_i w_{Aj_i} w_{Bj_i} s_x \sum_i w_{Ak_i} w_{Bk_i} \\
&= s_x^2 (\sum_i w_{A_i} w_{B_i})^2 \quad \text{assuming weights are antenna-independent}
\end{aligned} \tag{11}$$

The correlation coefficient R_{AB} between visibilities from beams A and B is

$$\begin{aligned}
R_{AB} &= \frac{\mathrm{Cov}[V_{Ajk}, V_{Bjk}]}{\sqrt{\mathrm{Var}[V_{Ajk}] \cdot \mathrm{Var}[V_{Bjk}]}} \\
&= \frac{s_x^2 (\sum_i w_{A_i} w_{B_i})^2}{\sqrt{s_x^2 (\sum_i w_{A_i}^2)^2 s_x^2 (\sum_i w_{B_i}^2)^2}} \\
&= \frac{(\sum_i w_{A_i} w_{B_i})^2}{\sum_i w_{A_i}^2 \sum_i w_{B_i}^2}
\end{aligned} \tag{12}$$

A.1 The F ratio

Heywood defines the ratio F to be $F = \frac{\sigma_c}{\sigma_i}$: c for combined, i for independent.

Let X,Y be the two signals to be combined. For the i combination, $\mathrm{Cov}[X, Y] = 0$. Then

$$
\begin{aligned}
F &= \sqrt{\frac{\mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X, Y]}{\mathrm{Var}[X] + \mathrm{Var}[Y]}} \\
&= \sqrt{1 + \frac{2\,\mathrm{Cov}[X, Y]}{\mathrm{Var}[X] + \mathrm{Var}[Y]}} \\
&= \sqrt{1 + \frac{2r[X, Y]\sigma_X\sigma_Y}{\sigma_X^2 + \sigma_Y^2}}
\end{aligned}
$$

and if $\sigma_X = \sigma_Y = \sigma$

$$
F = \sqrt{1 + r[X, Y]} \tag{13}
$$

A.2 Statistical identities used

$$
\begin{aligned}
\mathrm{Var}[X + Y] &= \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}[X, Y] && (14) \\
\mathrm{Var}[cX] &= c^2\,\mathrm{Var}[X] && (15) \\
\mathrm{Cov}[X + Y, Z] &= \mathrm{Cov}[X, Z] + \mathrm{Cov}[Y, Z] && (16) \\
\mathrm{E}[XY] &= \mathrm{E}[X]\,\mathrm{E}[Y] + \mathrm{Cov}[X, Y] && (17) \\
\mathrm{Var}[XY] &= \mathrm{E}[X^2Y^2] - \mathrm{E}[XY]^2 \\
&= \mathrm{Cov}[X^2, Y^2] \\
&\quad + [\mathrm{Var}[X] + \mathrm{E}[X]^2][\mathrm{Var}[Y] + \mathrm{E}[Y]^2]] \\
&\quad - [\mathrm{Cov}[X, Y] + \mathrm{E}[X]\,\mathrm{E}[Y]]^2 && (18)
\end{aligned}
$$

Correlation coefficient $r[X, Y]$:

$$
r[X, Y] = \frac{\mathrm{Cov}[X, Y]}{\sigma_X\sigma_Y} \tag{19}
$$

References

Heywood, I., Serra, P., Hotan, A., & McConnell, D. 2016, ACES Memo #13 Quantifying and mitigating correlated noise between formed beams on the ASKAP Phased Array Feeds, http://www.atnf.csiro.au/projects/askap/ACES-memos

Jeffs, B. D., Warnick, K. F., Landon, J., Waldron, J., Jones, D., Fisher, J. R., & Norrod, R. D. 2008, ISTSP, 2, 635

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