An introduction to Weak Scattering

or

Why do stars twinkle?

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Co-learnium
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Weak scintillation

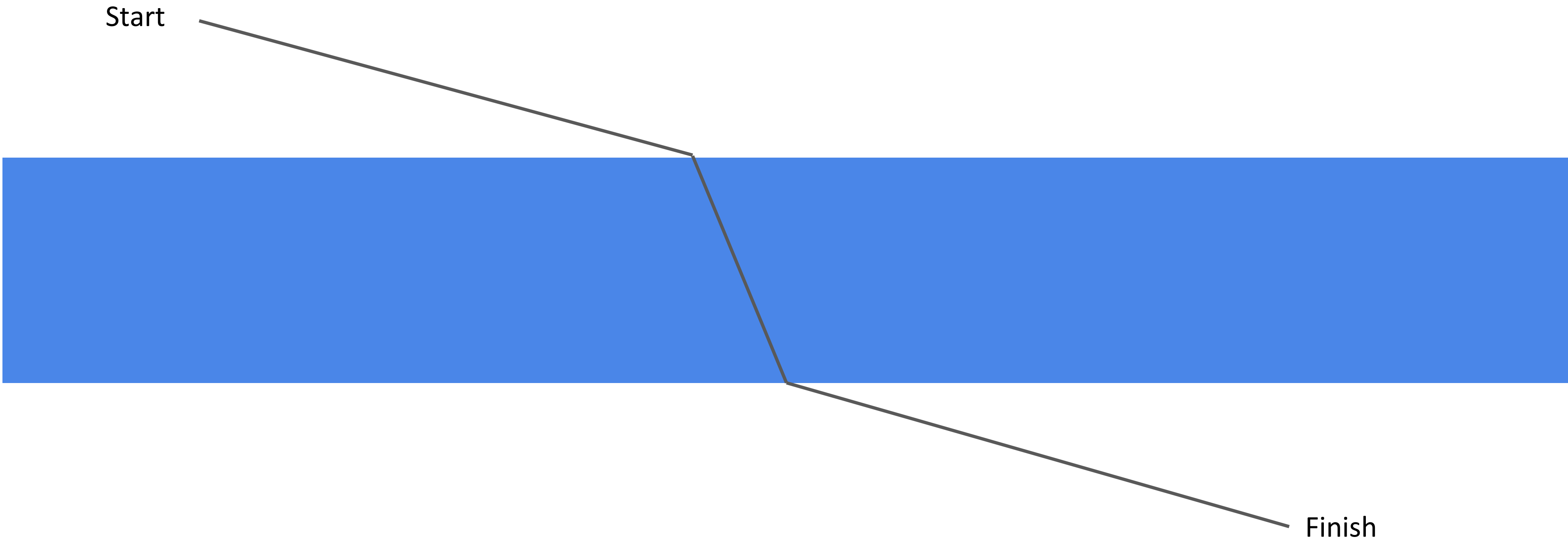
Optical (naked eye)

- Twinkling stars

Radio (particularly at lower frequencies)

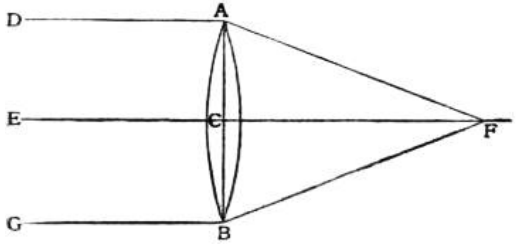
- Ionospheric scintillation
- Interplanetary scintillation
- Interstellar scintillation

Principle of least time (Fermat’s principle)



Principle of least time

We can take this principle as a starting point to understand how simple optical components work.



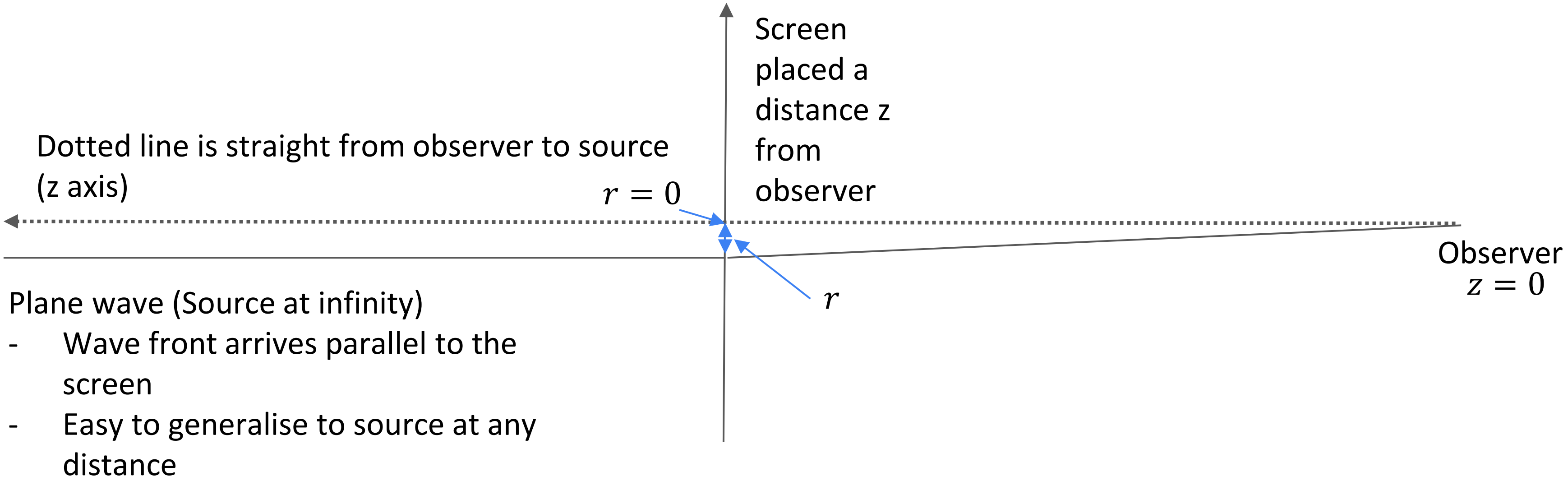
But the principle usually prompts most people to ask *how* light is able to

1. "solve" the problem of finding the quickest path
2. Do it at the speed of light

There is only really one possible way, and that is to explore every possible path in parallel.

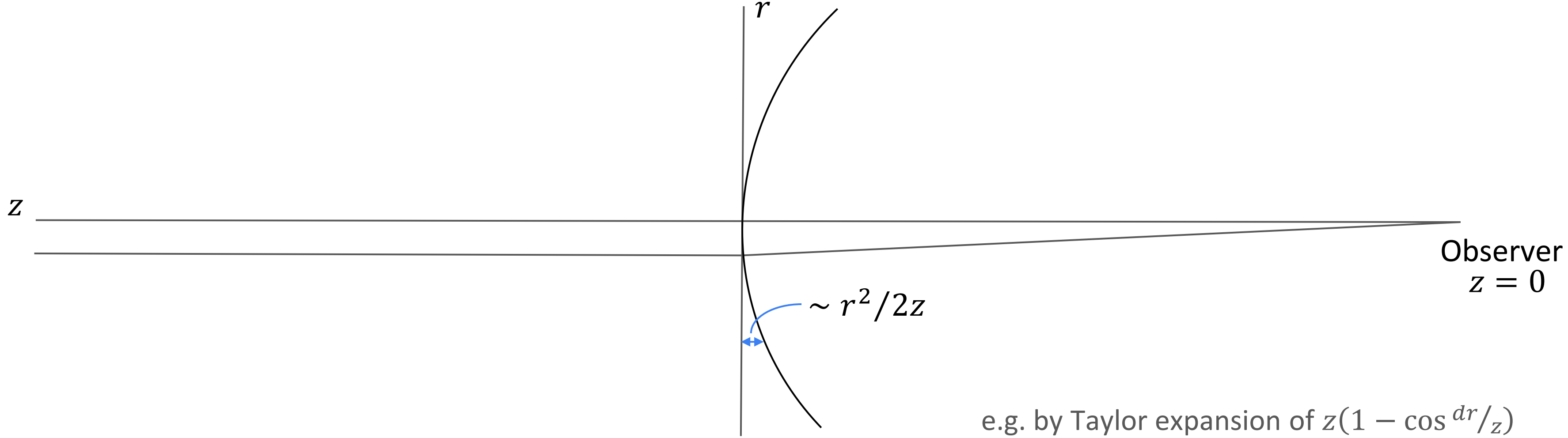
Thin screen.

- Take the simpler case of propagation in free space (i.e. no phase screen – we'll add that later)
- Examine very small deviations from the quickest path



Thin screen

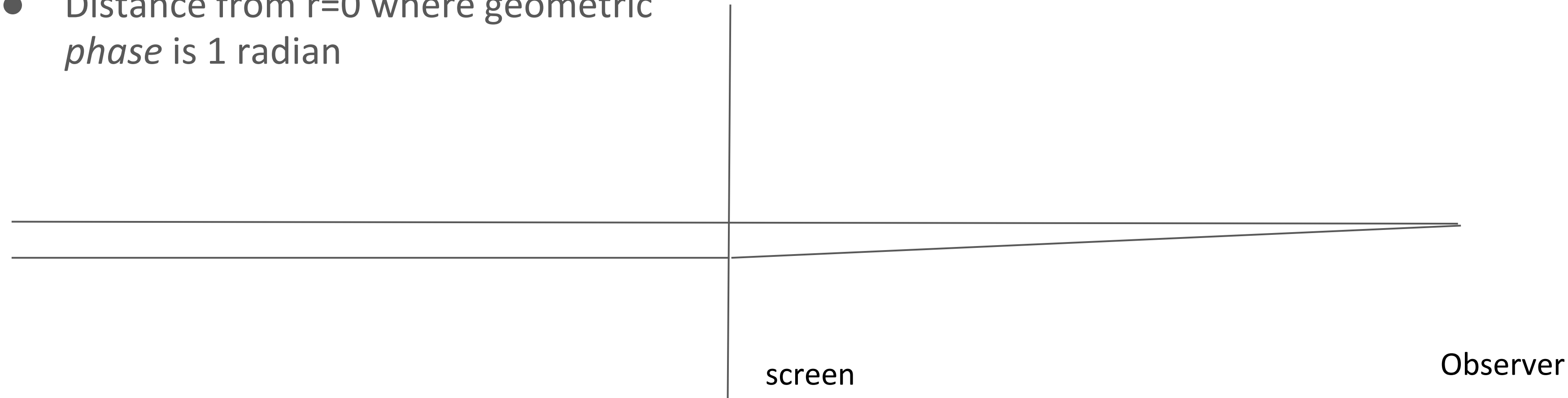
- Shortest path is obviously a straight line for free space
- Paths that deviate slightly by $\pm r$ at the screen arrive at the observer having travelled $\sim r^2/2z$ further.
- ***Quadratically*** increasing phase for paths further away from straight line



e.g. by Taylor expansion of $z(1 - \cos {}^{dr}/_{z})$

Thin screen

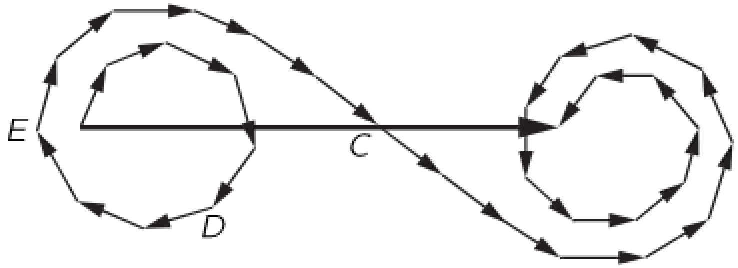
- **Rate of change of phase is what makes the point on the direct line of sight unique**
- Except for a very small path on the screen close to the direct path, light from any other small patch on the screen will arrive incoherently and will (almost) cancel out.
- **Fresnel scale** $\boldsymbol{r_F} = \sqrt{\frac{z\lambda}{2\pi}}$ Characterises the size of this special region[†]
- Distance from r=0 where geometric *phase* is 1 radian



[†]NB 2π not always included; see Wikipedia for form with source not at infinity

Fresnel-Kirchoff Integral

Vector sum of all points along the screen gives received signal (amplitude and phase)



Area near C (region where phase is in range$(-\pi, \pi)$) is first Fresnel zone

Euler Spiral From Feynman lectures in physics (Vol 1, ch 26)

Fresnel-Kirchoff Integral

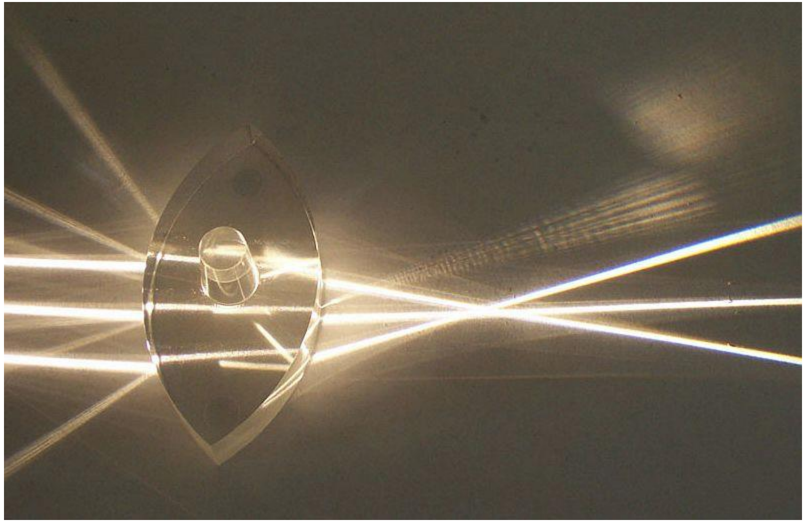
By placing slits and **phase screens** on our thin screen we can use this simple picture to explain

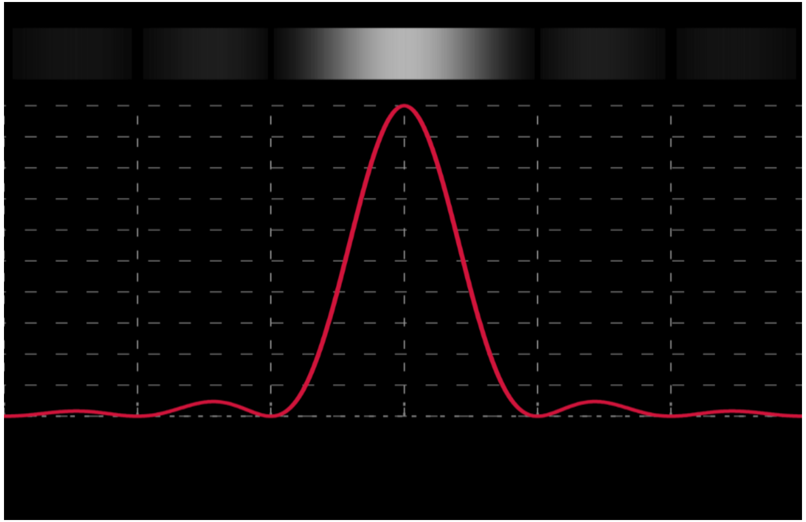
- Lenses (converging and diverging)
- diffraction
- Weak scintillation

"Weak" in this context means that phase fluctuations are *always* small compared with geometric phase:

$$r_F \ll r_{\text{diff}}$$

r_{diff} is the distance over which the phase of the screen changes by 1 radian (big r_{diff} means *weak* scatter)





Please see [this github gist](https://gist.github.com/johnsmorgan/6821781fccd6b02e0aaeec3d08c0e313)

`https://gist.github.com/johnsmorgan/6821781fccd6b02e0aaeec3d08c0e313`

You will need to download and run the notebook to see the plots

Aside (Fourier Transform relationship between aperture and PSF)

The Fresnel-Kirchoff integral resembles a Fourier Transform

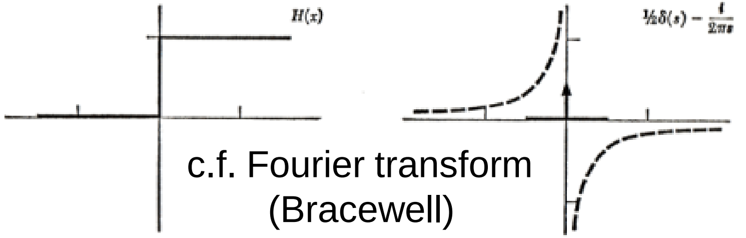
In fact, it can be reduced to a Fourier Tranform if two conditions are met (details in this lecture)

1. $D/z \ll 1$ (where D is greatest dimension of the aperture, z is distance to screen)
2. $D^2/z \ll \lambda$

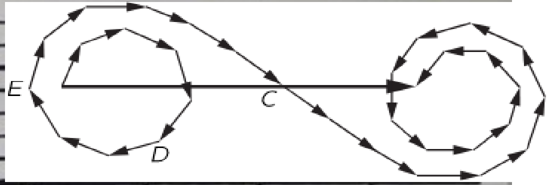
Neither applies in far-field (most astronomy)

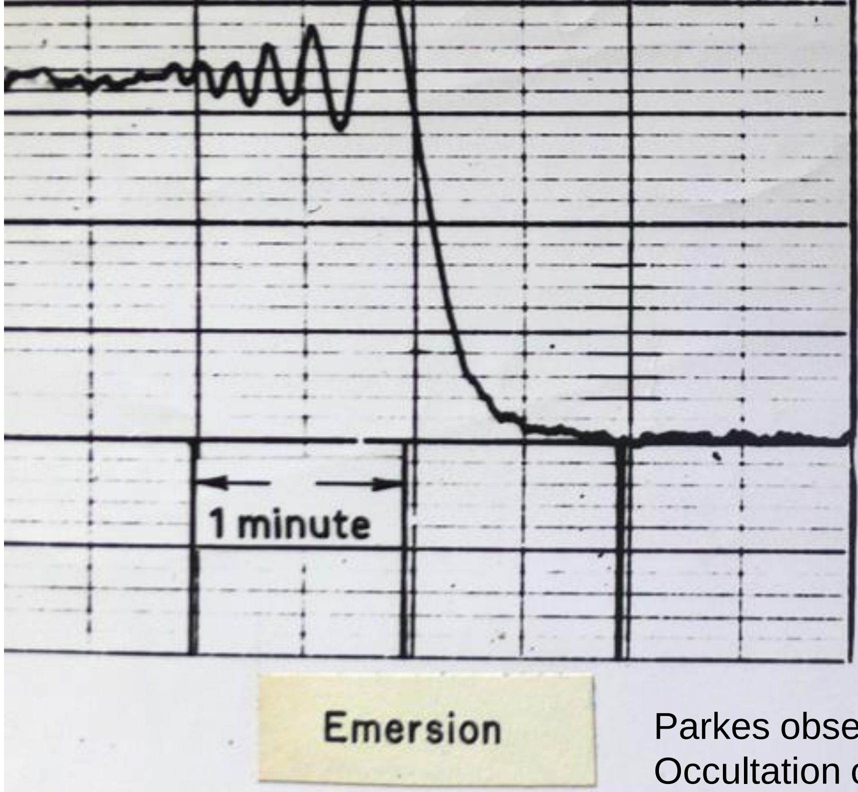
As well as near-field problems, there are also other simple optical systems that can't fulfil these requirements.

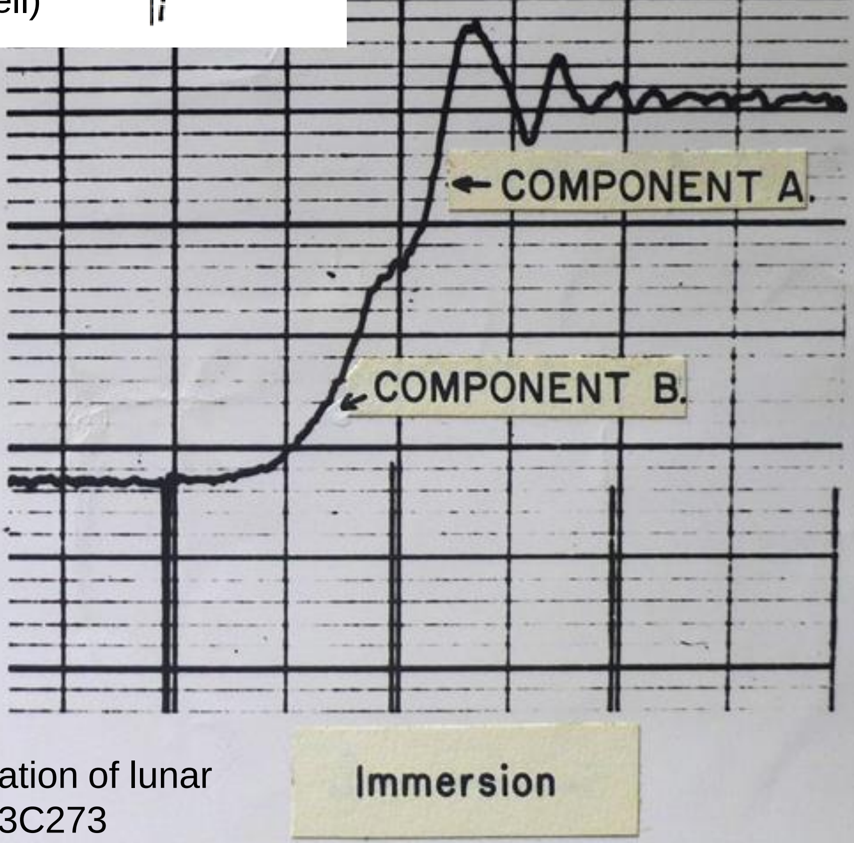
- We have to do the full integral
- However, it's pretty intuitive



c.f. Fourier transform
(Bracewell)







Parkes observation of lunar
Occultation of 3C273

Fresnel scale (and diffractive scale) examples (from Narayan)

Table 1. *Examples of scattering media in astronomy, with typical values of* λ, D, r_F, *and* r_{diff}

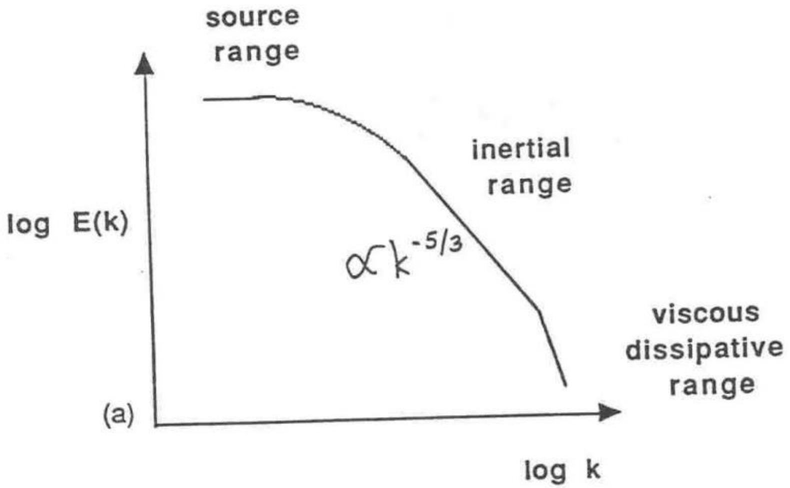
medium	λ / cm	D / cm	r_F / cm	r_{diff} / cm	régime of scattering: weak	régime of scattering: strong
optical						
Earth's atmosphere	5×10^{-5}	10^{6}	3	10	mostly	near horizon
planetary atmospheres[a]	10^{-4}	10^{14}	4×10^{4}	$10^{2} - 10^{6}$	early in occultation	deep in occultation
radio						
troposphere	20	10^{5}	6×10^{2}	$\sim 10^{5}$	yes	no
ionosphere	3×10^{2}	3×10^{7}	4×10^{4}	$\sim 10^{5}$	yes	sometimes
solar wind	10^{2}	10^{13}	10^{7}	$> 10^{7}$	mostly	close to the Sun
interstellar medium	10^{2}	10^{21}	10^{11}	$\sim 10^{9}$	no	yes

[a] Stars scintillate due to scattering in planetary atmospheres during occultations. The scattering is initially weak but becomes strong deep in the occultation (cf. Narayan & Hubbard 1988).

Turbulence

Turbulence deserves its own lecture! But briefly

- Turbulence is random but can be described statistically
    - Structure function
    - Power spectrum
- Larger scales dominate
- Inertial range can cover many orders of magnitude



From this lecture.

[Cronyn (1972)]

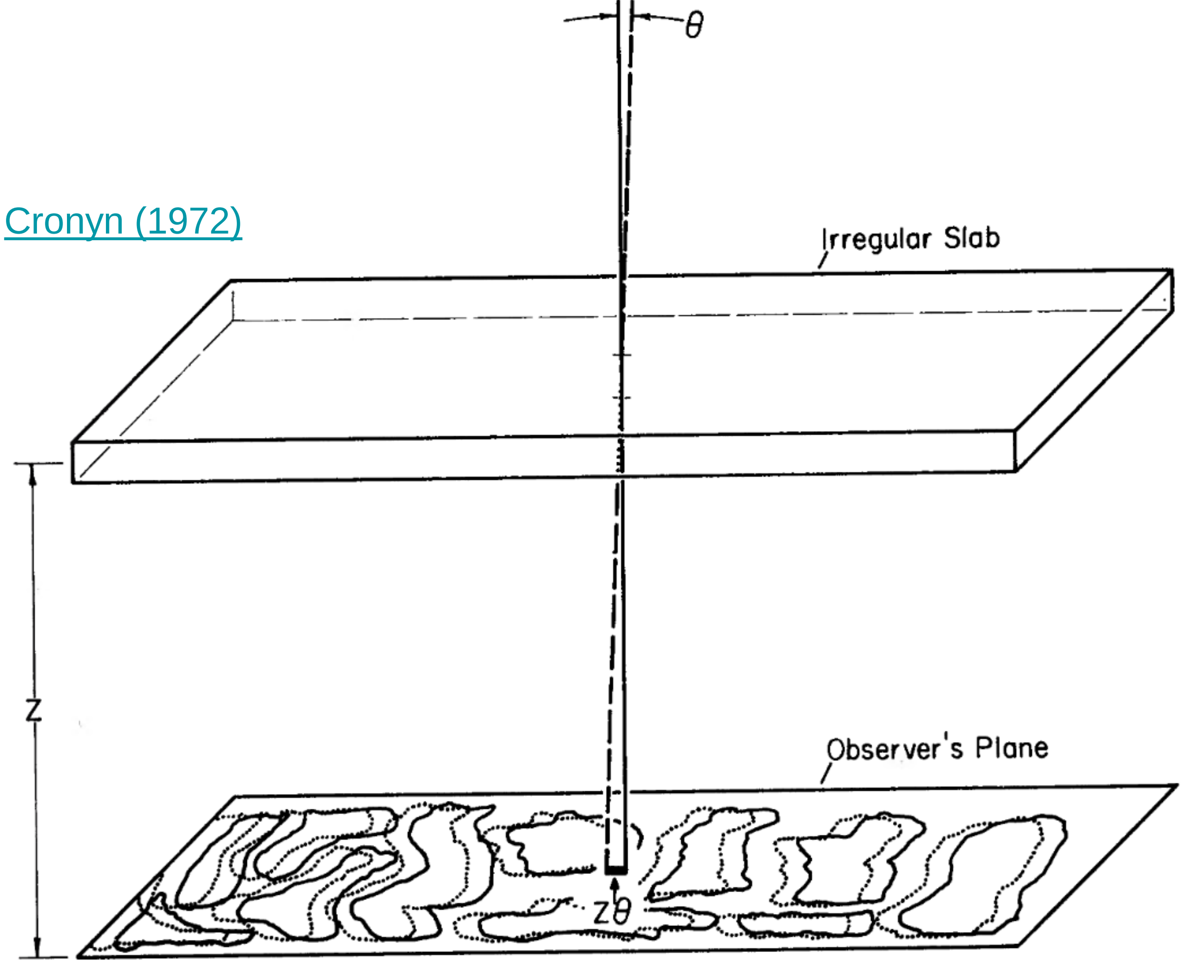
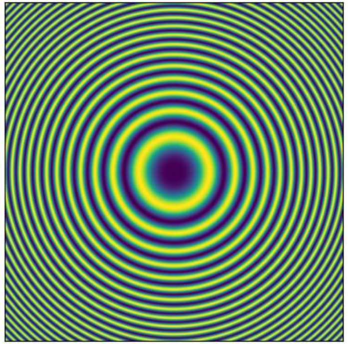


FIG. 3.—Pictorial idealization of diffraction patterns cast by two point sources separated by angle θ. Patterns are shifted with respect to each other by a distance $z\theta$, thereby smearing pattern structure finer than $z\theta$.

- Weak scintillation actually produces a diffraction pattern on the ground (characteristic size of patches is r_F)
- Velocity of the screen converts from spatial coordinates to a temporal power spectrum
- Finite source size (or large aperture for observing instrument) is a convolution
- Scattering is two dimensional!

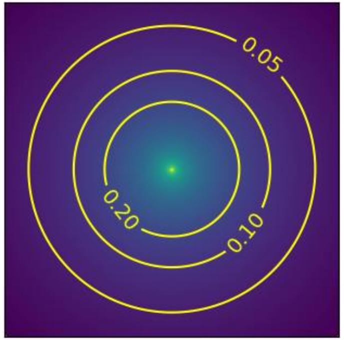
Into two dimensions

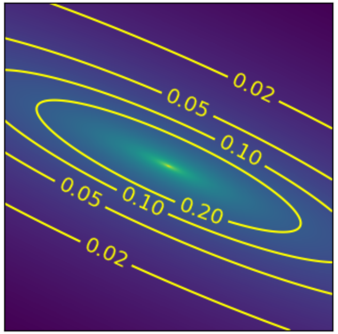


Fresnel filter

$$f(r) = \sin^2 r^2$$

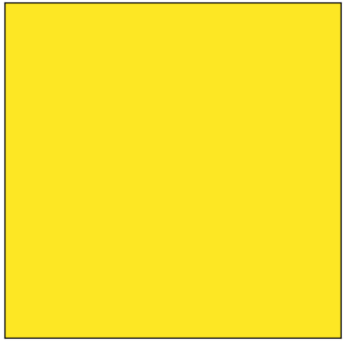
×

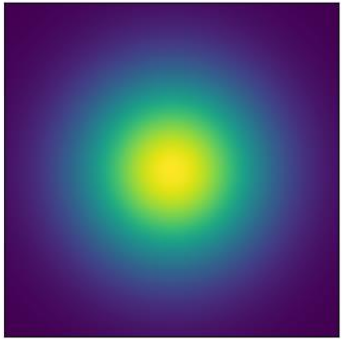


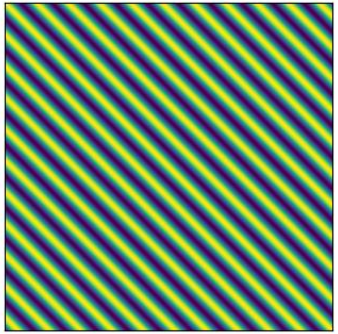


turbulence
power spectrum

×

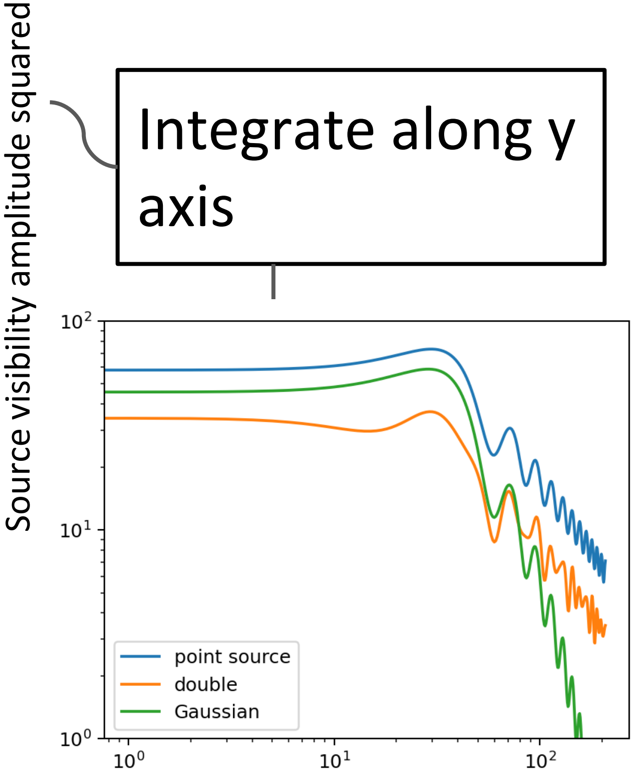






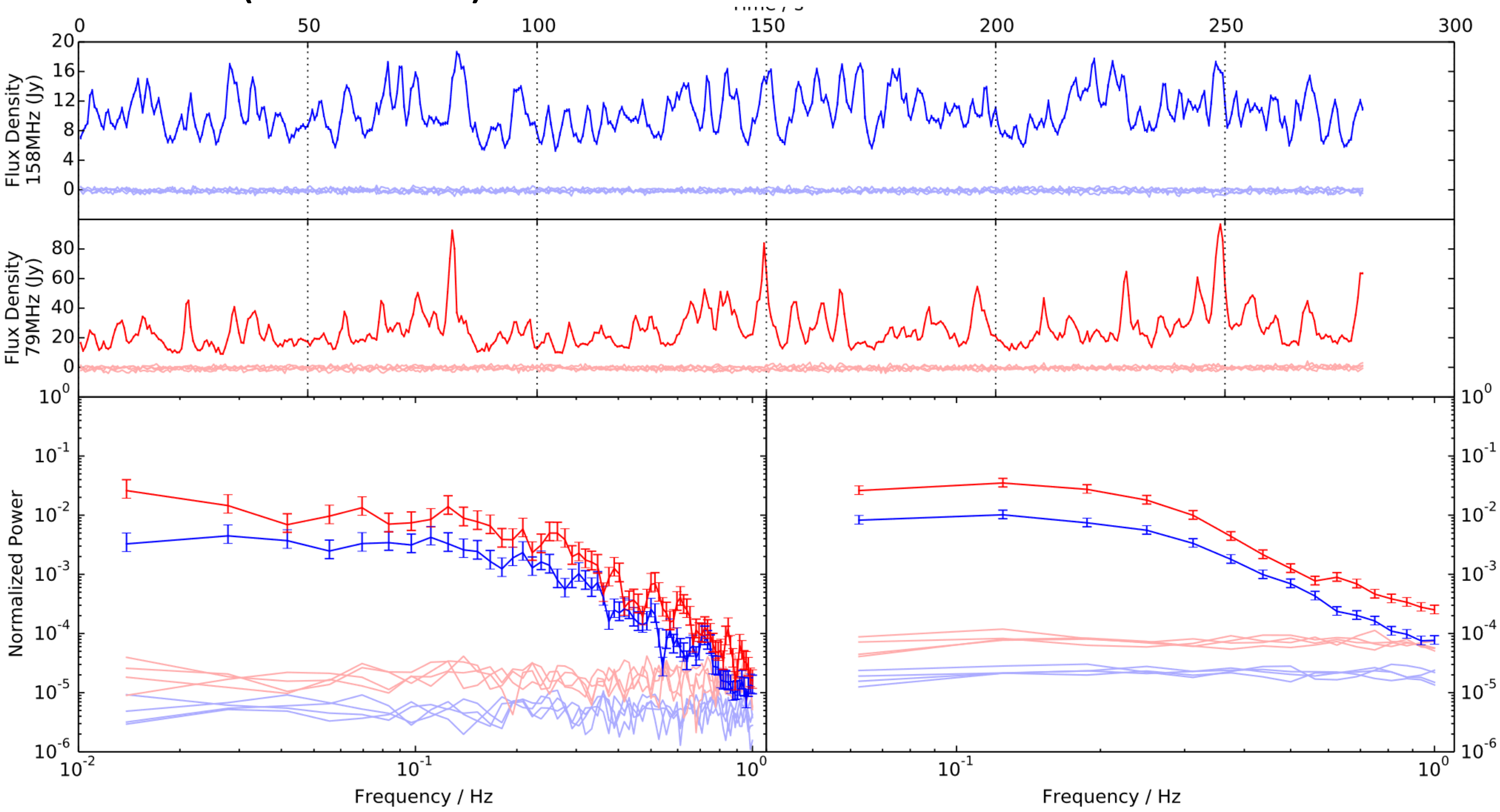
Axes are defined by the velocity vector

Integrate along y axis



All are odd or even functions but plotted with origin at centre for clarity

Real data (MWA IPS)



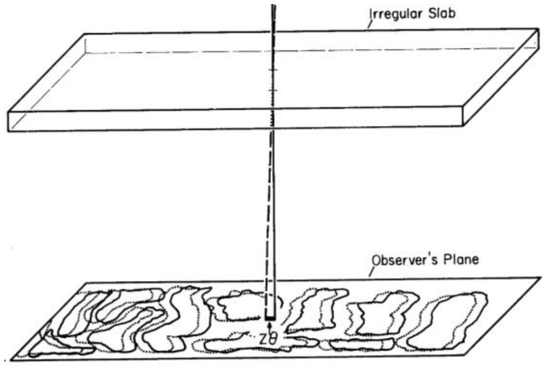
Transition into strong scattering

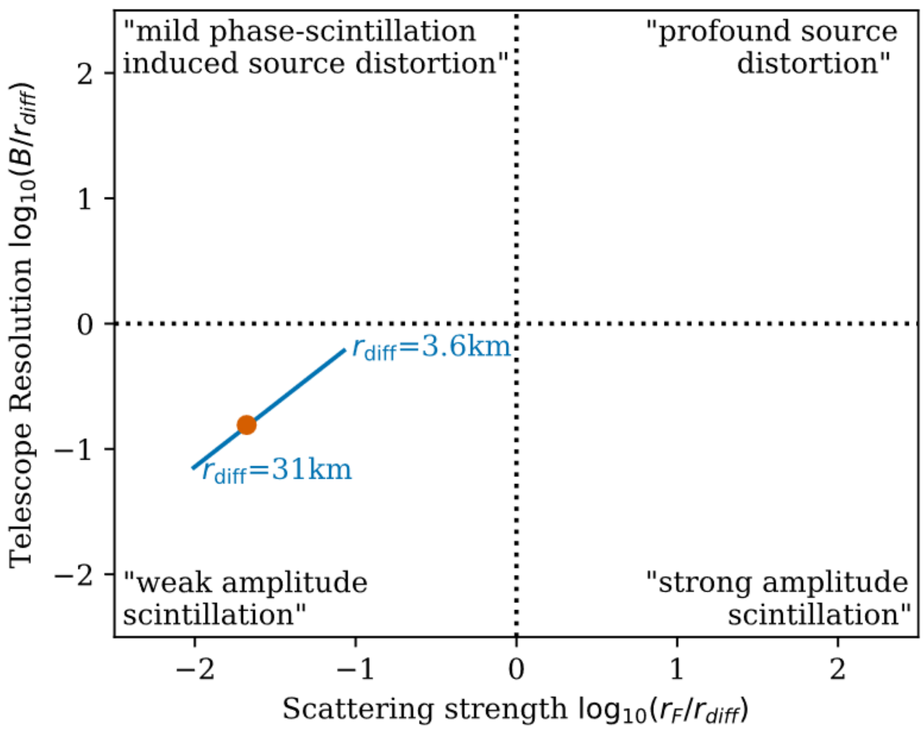
- Weak scintillation picture we have discussed is only valid when $r_F \ll r_{\text{diff}}$
- In this case, focal length of ‘lens’ is guaranteed to be $\gg z$ (= weak magnification)
- As r_{diff} approaches r_F we get strong focussing: big peaks in intensity
- c.f. gravitational lensing terminology
- As r_{diff} becomes much smaller than r_F the physics is rather different (a topic for another lecture)

Finite sources and finite apertures

Both are similar:

- Roughly speaking, both can be thought of as a convolution in the time domain
  - = multiplication in power spectrum
  - Suppression of amplitude scintillation
- Each r_F-sized patch of the source scintillates somewhat independently
- A large aperture smooths out weak amplitude scintillations in a similar way *but* remember that phase is affected also





Waszewski+ 2022

Figure 1. Scattering strength vs baseline length in terms of scattering scale, following Cornwell et al. (1989). Descriptions of asymptotic regimes are also from Cornwell et al. (though we adopt our definition of r_F; see Equation (7)). The blue line is for the range of $r_{\rm diff}$ observed by Mevius et al. (2016), but scaled to our observing frequency of 154 MHz. Baseline length B is assumed to be 2.2 km (MWA Phase I). The orange point is calculated from the scintillation index observed by Morgan & Ekers (2021) with the MWA at 162 MHz. Height of the ionosphere is assumed to be 300 km.

Summary of weak scatter

- Power spectrum of turbulence (typically negative power law form)
- Multiply by Fresnel filter and integrate along y axis
- **Velocity of the screen converts from spatial units to temporal power spectrum**

We have assumed weak scattering: $r_F \ll r_{\mathrm{diff}}$

- Consider source structure unless $\theta_{\mathrm{src}} < \theta_F$ (where $\theta_F = r_F / z$)
  - If $\theta_{\mathrm{src}} \gg \theta_F$ and no small-scale structure at all – no scintillation!
- Consider aperture of observing instrument unless ($D < r_{\mathrm{diff}}$)

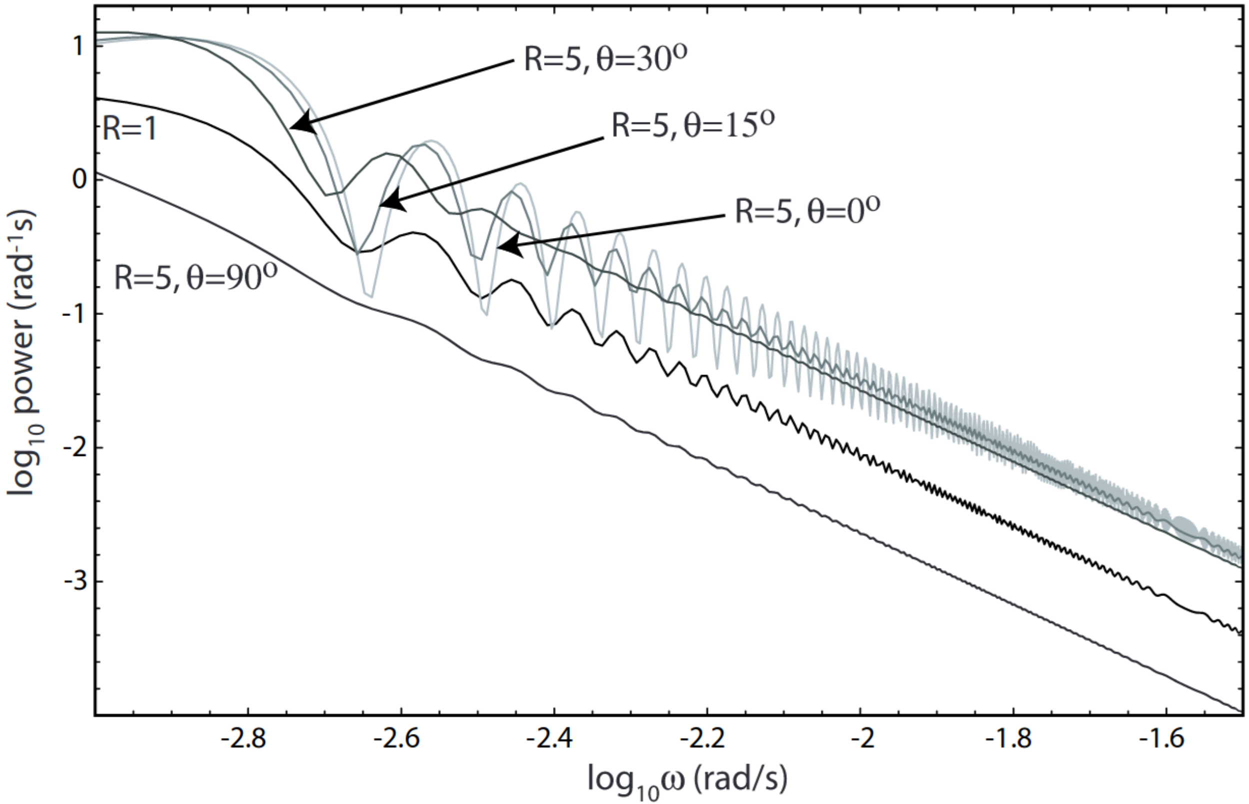
Additional Slides

$$P(\omega) = \frac{1}{v_{\rm ISS}} \int_{-\infty}^{\infty} {\rm d}q_y \, P_{\rm pt}\left(\frac{\omega}{v_{\rm ISS}}, q_y\right) \left| V\left(\frac{\omega z}{v_{\rm ISS} k}, \frac{q_y z}{k}\right)\right|^2, \qquad (1)$$

where $V(\boldsymbol{r})$ is the source visibility measured on a baseline $\boldsymbol{r}$, $k = 2\pi/\lambda$ is the wavenumber, $v_{\rm ISS}$ is the scintillation velocity, here oriented along the x-axis, and

$$P_{\rm pt}(\boldsymbol{q}) = 8\pi r_{\rm e}^2 \lambda^2 \Phi_{N_{\rm e}}(\boldsymbol{q}) \sin^2\left(\frac{q^2 z}{2k}\right) \qquad (2)$$

Macquart & de Bruyn (2007)



Fresnel filter minima can become much more prominent for high anisotropy with structure elongated in direction perpendicular to velocity.

Inner scale cuts of high end of power spectrum for IPS

Figure 3. Scintillation power spectra for a point source of unit flux density with $v_{\rm ISS} = 50\,{\rm km\,s^{-1}}$ and $z = 10\,{\rm pc}$ for various R and θ. Macquart & de Bruyn (2007)

Resources

- Narayan (1992) (pdf available on request)
- J-P's lecture notes (pdf available on request)
- Thompson, Moran & Swenson (available online)
- Born & Wolf (in Curtin Library)
- See also Feynman Lectures Vol 1 Ch 26
- Development of weak scattering scintillation theory can be found starting with these papers (Paper I, Paper II). See also references in Narayan (1992).
- See this lecture for a great introduction to turbulence