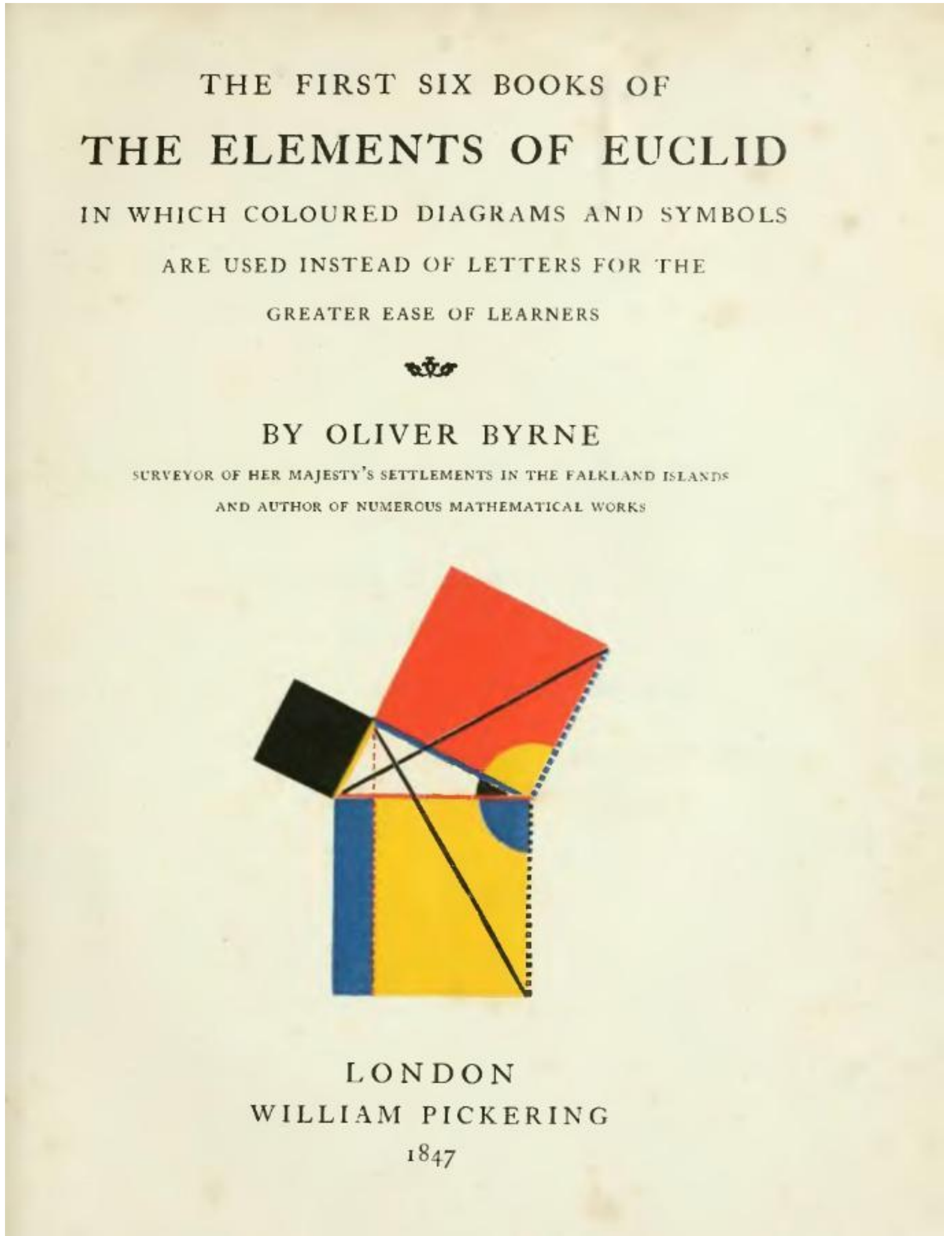
Strange new worlds

A brief history of Euclidean and Non-Euclidean Space


THE FIRST SIX BOOKS OF

THE ELEMENTS OF EUCLID

IN WHICH COLOURED DIAGRAMS AND SYMBOLS ARE USED INSTEAD OF LETTERS FOR THE GREATER EASE OF LEARNERS

BY OLIVER BYRNE

SURVEYOR OF HER MAJESTY'S SETTLEMENTS IN THE FALKLAND ISLANDS AND AUTHOR OF NUMEROUS MATHEMATICAL WORKS

LONDON
WILLIAM PICKERING
1847

Editions [edit]

- 1460s, Regiomontanus (incomplete)
- 1482, Erhard Ratdolt (Venice), first printed edition[30]
- 1533, *editio princeps* by Simon Grynäus
- 1557, by Jean Magnien and Pierre de Montdoré (fr), reviewed by Stephanus Gracilis (only propositio
- 1572, Commandinus Latin edition
- 1574, Christoph Clavius

Translations [edit]

- 1505, Bartolomeo Zamberti (de) (Latin)
- 1543, Niccolò Tartaglia (Italian)
- 1557, Jean Magnien and Pierre de Montdoré, reviewed by Stephanus Gracilis (Greek to Latin)
- 1558, Johann Scheubel (German)
- 1562, Jacob Kündig (German)
- 1562, Wilhelm Holtzmann (German)
- 1564–1566, Pierre Forcadel (fr) de Béziers (French)
- 1570, Henry Billingsley (English)
- 1572, Commandinus (Latin)
- 1575, Commandinus (Italian)
- 1576, Rodrigo de Zamorano (Spanish)
- 1594, Typographia Medicea (edition of the Arabic translation of Nasir al-Din al-Tusi)
- 1604, Jean Errard (fr) de Bar-le-Duc (French)
- 1606, Jan Pieterszoon Dou (Dutch)
- 1607, Matteo Ricci, Xu Guangqi (Chinese)
- 1613, Pietro Cataldi (Italian)
- 1615, Denis Henrion (French)
- 1617, Frans van Schooten (Dutch)
- 1637, L. Carduchi (Spanish)
- 1639, Pierre Hérigone (French)
- 1651, Heinrich Hoffmann (German)
- 1651, Thomas Rudd (English)
- 1660, Isaac Barrow (English)
- 1661, John Leeke and Geo. Serle (English)
- 1663, Domenico Magni (Italian from Latin)
- 1672, Claude François Milliet Dechales (French)
- 1680, Vitale Giordano (Italian)
- 1685, William Halifax (English)
- 1689, Jacob Knesa (Spanish)
- 1690, Vincenzo Viviani (Italian)
- 1694, Ant. Ernst Burkh v. Pirckenstein (German)
- 1695, C. J. Vooght (Dutch)
- 1697, Samuel Reyher (German)
- 1702, Hendrik Coets (Dutch)
- 1705, Charles Scarborough (English)
- 1708, John Keill (English)
- 1714, Chr. Schessler (German)
- 1714, W. Whiston (English)
- 1720s Jagannatha Samrat (Sanskrit, based on the Arabic translation of Nasir al-Din al-Tusi)[31]
- 1731, Guido Grandi (abbreviation to Italian)

Euclid's Elements

Euclid starts with a series of simple axioms

- definition of a line as shortest distance between two points)

and uses these a whole series of theorems

- Angles in a triangle add up to 180°
- Circumference of a circle $2\pi r$
- Pythogoras' theorem

Every school lesson in geometry can be traced back to this textbook

Held up as a model of logic for centuries

Impact on western thought

We hold these truths to be self-evident, that all men are created equal, that they are endowed by their Creator with certain unalienable Rights, that among these are Life, Liberty and the pursuit of Happiness.

POSTULATES.

Let it be granted,

1. That a straight line may be drawn from any one point to any other point:

2. That a terminated straight line may be produced to any length in a straight line:

3. And that a circle may be described from any centre, at any distance from that centre.

[6]

AXIOMS.

1. Things which are equal to the same thing are equal to one another.

2. If equals be added to equals the wholes are equal.

3. If equals be taken from equals the remainders are equal.

4. If equals be added to unequals the wholes are unequal.

5. If equals be taken from unequals the remainders are unequal.

6. Things which are double of the same thing are equal to one another.

7. Things which are halves of the same thing are equal to one another.

8. Magnitudes which coincide with one another, that is, which exactly fill the same space, are equal to one another.

9. The whole is greater than its part.

10. Two straight lines cannot enclose a space.

11. All right angles are equal to one another.

12. If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles.

Are the foundations valid?

All can be demonstrated on a small piece of paper with a compass and straightedge apart from two:-

- Definition of parallel lines
- Postulate 5

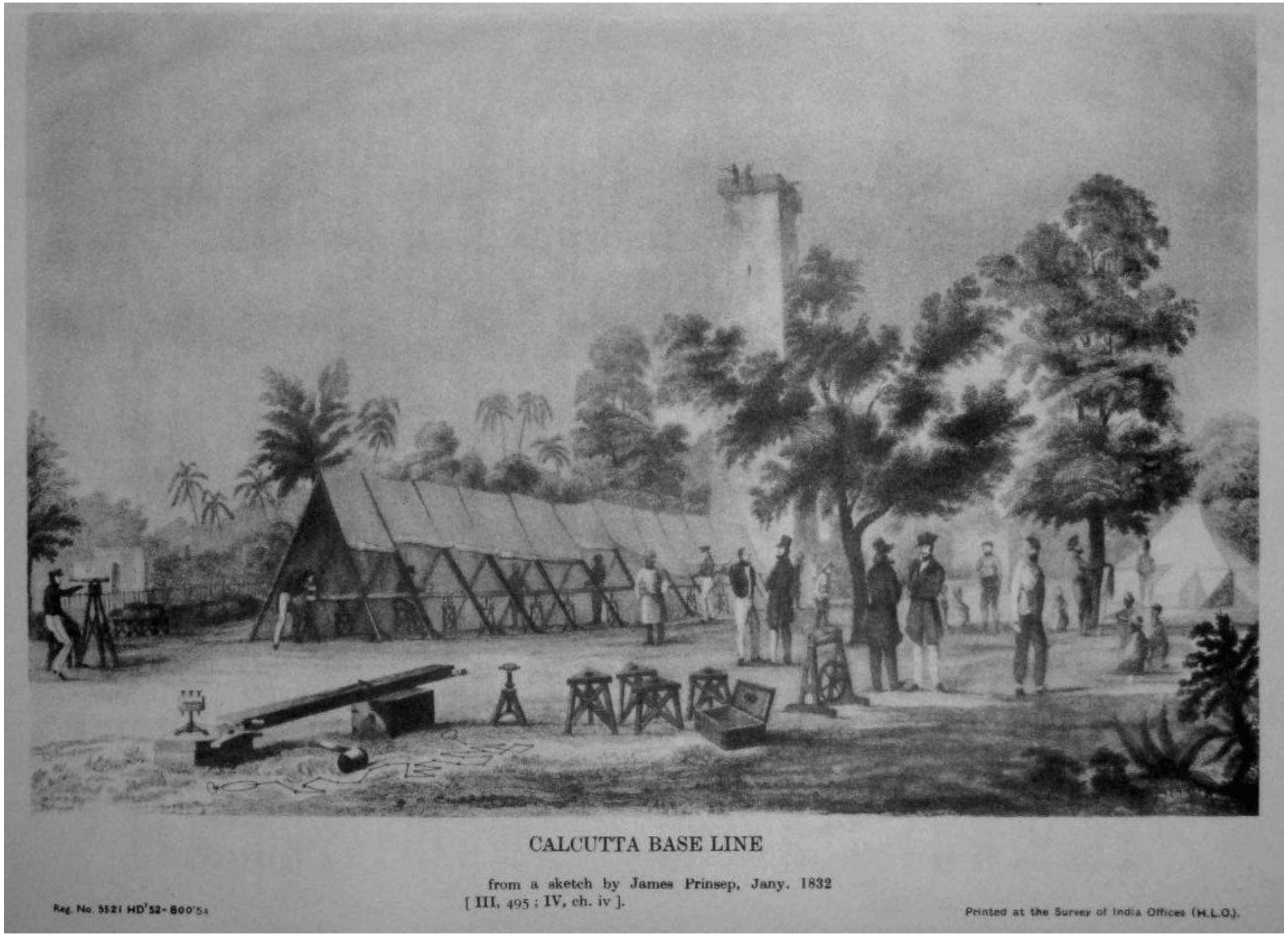
  That, if a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

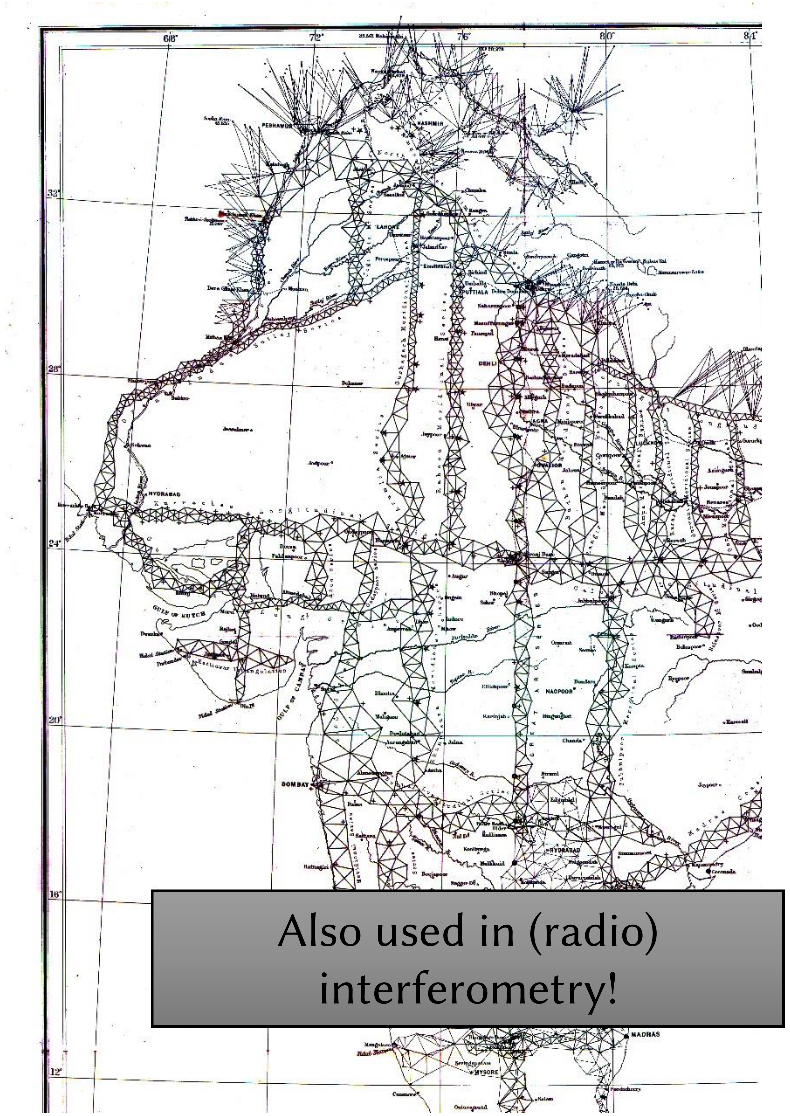
Proving the 5th postulate

Has been attempted for millennia but no-one succeeded.

- Dozens of almost equivalent postulates have been proven
  - *In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point. (Playfair's axiom)*
- In the early 1800s, a number of mathematicians (Bolyai, Lobachevskii, Gauss) began to explore the consequences if the 5th postulate were not true
  - Now known as "Non-Euclidean Geometry"
  - Possible Gauss may even have tested it when surveying the Kingdom of Hanover

Great Trigonometric Survey (of India)





Also used in (radio) interferometry!

Non-Euclidean Geometry

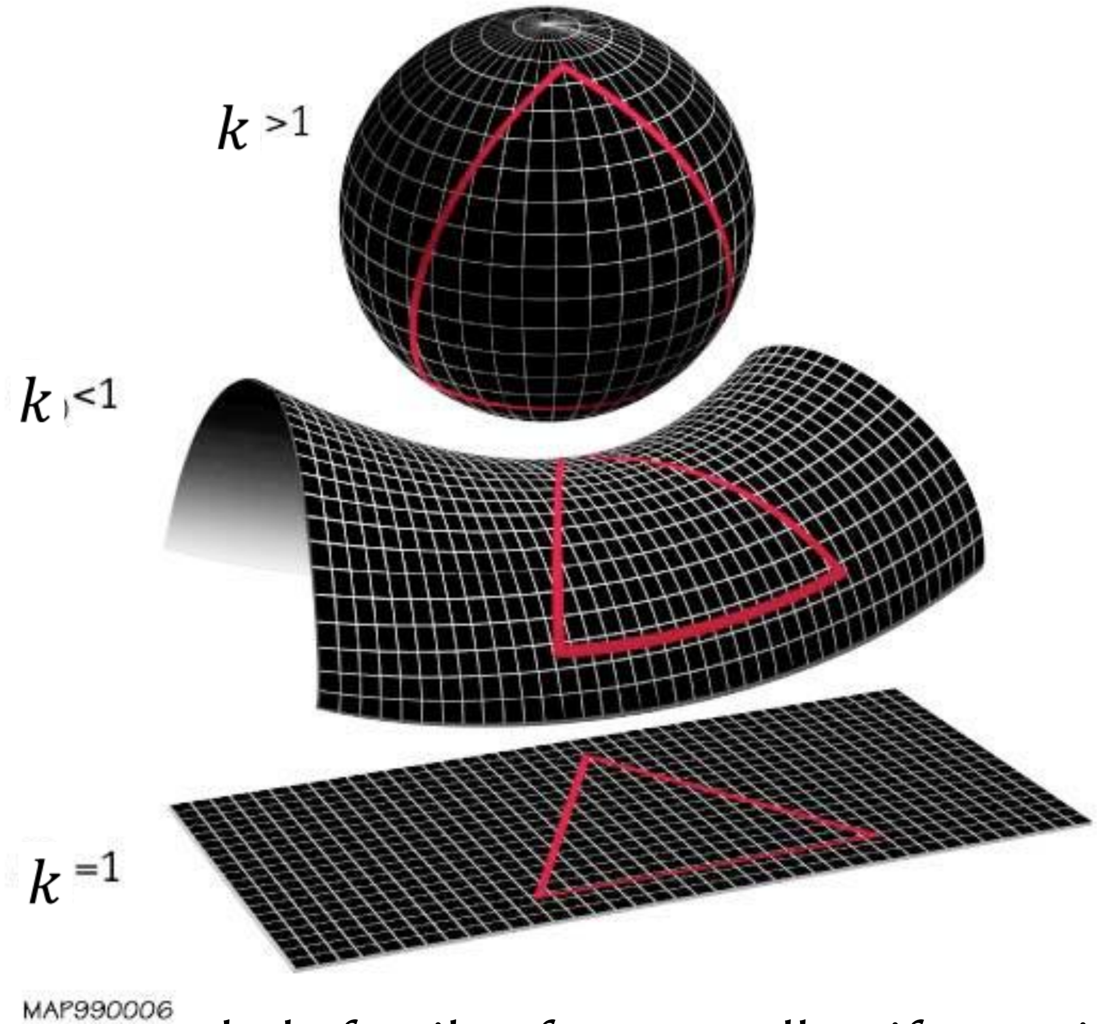
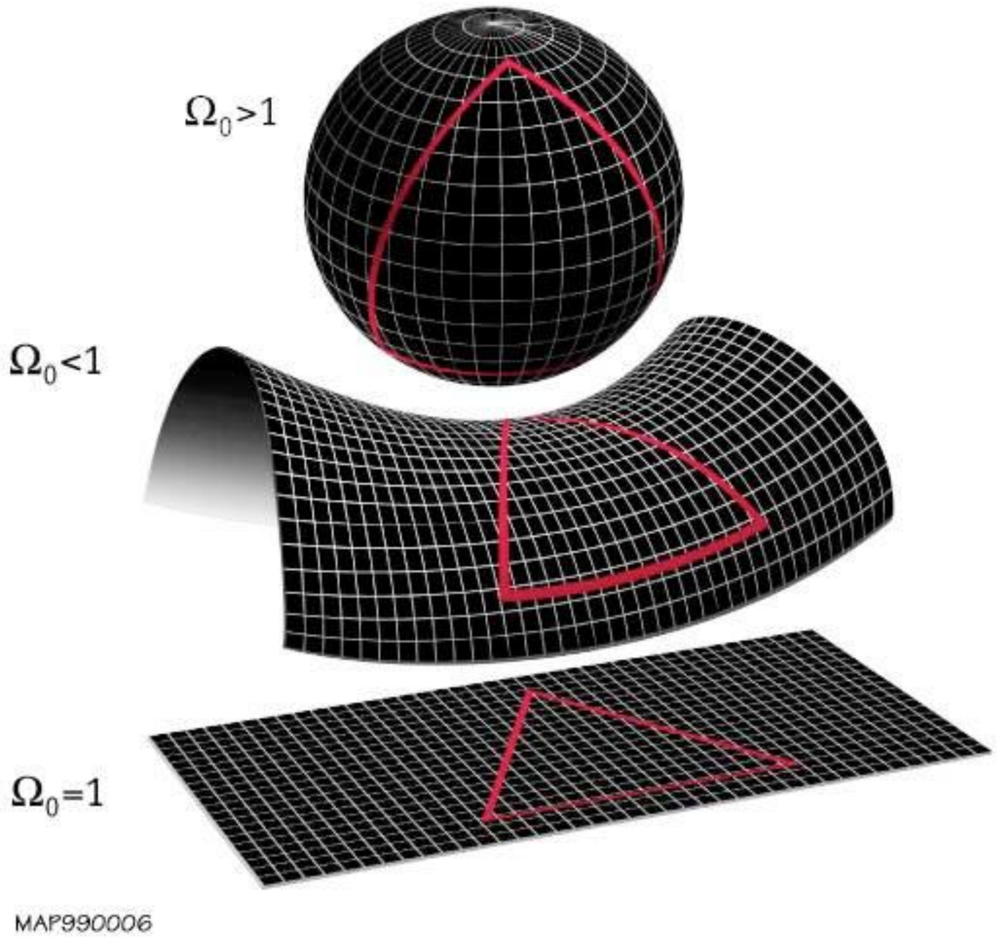


Image credit: NASA

There are a whole family of spaces, all uniform with self-consistent geometry. All resemble flat space on sufficiently small scales

What these diagrams mean

- We can *represent* (imperfectly in the case of hypobolic space) 2D curved space in 3 dimensions.
- We would need 6 dimensions to represent 3D space and 10 to represent 4D (e.g. spacetime)
- But that's not the point
- Curvature is a property of the space itself – we are not proposing new dimensions



A Brief History of General Relativity

- 1915-1916 – Einstein publishes General Relativity
  - Eddington jokes that less than 3 people understand it
- 1916 Schwarzschild comes up with the first solution
- 1917 – Einstein, de Sitter and others solve GR for Homogeneous Universes
  - Necessitates Λ for static solution
- 1922-1927 – Friedmann and Lemaitre do the same for expanding Universes containing matter
- 1929 – Hubble publishes evidence that the Universe is expanding
- 1934 – Milne and McCrea reproduce Friedmann's equations using Newtonian dynamics and gravity!*

*and $E = mc^2$

The Friedman Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}$$

$a(t)$ is a scalar proportional to the size of the Universe

($\dot{a} > 0$ means expanding Universe)

In the Newtonian derivation, this is balanced against:-

$\frac{8\pi G\rho}{3}$ Gravitational force

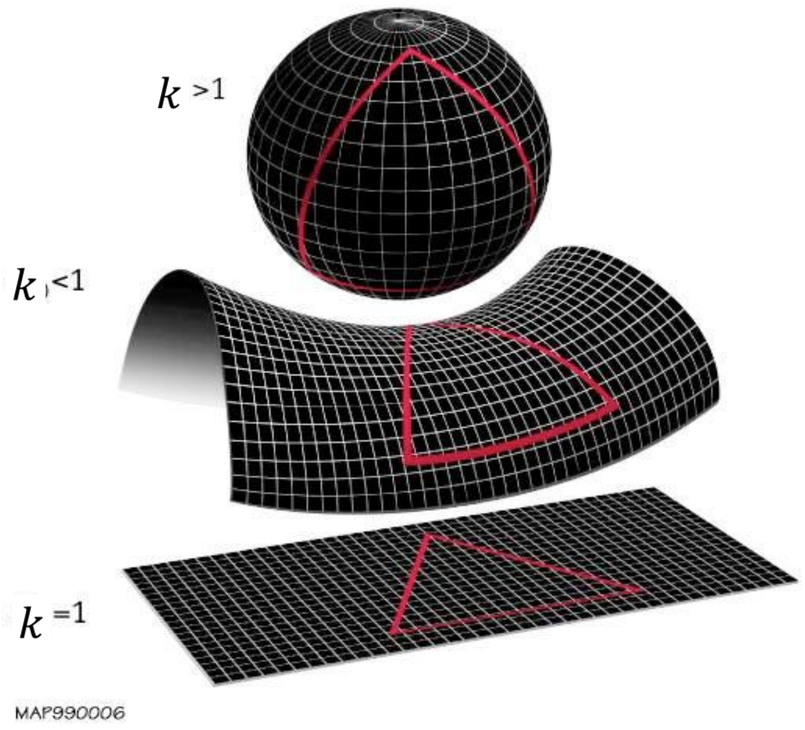
$\frac{-kc^2}{a^2}$ total energy (potential + kinetic)

Newtonian vs GR interpretation

Following Harrison (2000)

Newtonian	GR
Gravity is instantaneous	Gravity travels at *c*
Redshift due to relative motion	Redshift due to Universe expansion
Grav. Acceleration due to force	Straight lines in curved space time

k	Newtonian Interpretation	GR Geometry	Fate of Universe
-1	Hyperbolic "orbits"	Hyperbolic	Wimper
0	Parabolic "orbits"	Flat (Euclidean)	Wimper
1	Elliptical "orbits"	Spherical	Crunch



Why do they agree?

Is our visual system hyperbolic?

“Alleyway experiment”: subject helps to arrange lights in a darkroom into parallel lines. Lines of lights are not, in fact parallel, but obey Euclidean geometry. (first conducted by F. Hillebrand 1902)

Subjects will also rank images created using non-Euclidean geometries as more “realistic” than those created using Euclidean geometry.

Related to binocular vision? Human eye anatomy? Or “distance compression” a more “practical” metric (c.f. magnitudes, decibels, etc.)

Which other sense is hyperbolic?

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RESEARCH ARTICLE | NEUROSCIENCE

Hyperbolic geometry of the olfactory space

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Abstract

In the natural environment, the sense of smell, or olfaction, serves to detect toxins and judge nutritional content by taking advantage of the associations between compounds as they are created in biochemical reactions. This suggests that the nervous system can classify odors based on statistics of their co-occurrence within natural mixtures rather than from the chemical structures of the ligands themselves. We show that this statistical perspective makes it possible to map odors to

Perspectives

(a quarter of the way through the 21st C)

Edward Harrison (2000)

“The standard model of the Universe at the end of the nineteenth century was unlike the standard model at the end of the twentieth century in almost every respect. This prompts the question: Is it possible that the standard model of the Universe at the end of the twenty-first century will be totally unlike that at the end of the twentieth century? The Victorians were confident they were close to the truth”

Lahav & Liddle (2019)

“The concordance model is now well established, and there seems little room left for any dramatic revision of this paradigm. A measure of the strength of that statement is how difficult it has proven to formulate convincing alternatives.”

Extra slides

The Friedmann Equation

- The most important equation in cosmology!
- Describes the expansion of the Universe
- Newtonian approach (c.f. GR derivation)
- Find KE and PE of test particle of mass m in a uniform, expanding medium of density ρ
- Recall Gauss' theorem:
  - In a spherically symmetric mass distribution, a particle feels no force from material at greater radii
  - Force from material at smaller radii is as if all mass were concentrated at a point

The Friedmann Equation

- The most important equation in cosmology!
- Describes the expansion of the Universe
- Newtonian approach (c.f. GR derivation)
- Find KE and PE of test particle of mass m in a uniform, expanding medium of density ρ
- Universe has no centre so we are free to arbitrarily choose a reference point with no loss of generality

  - The whole of the Universe?
  - An infinitesimal part of the Universe

The Friedmann Equation

- Mass of material at smaller radii exerting gravitational force: $M = \frac{4}{3}\pi r^3 \rho$
- Gravitational force on test particle:

$$F = \frac{GMm}{r^2} = \frac{4\pi G\rho r m}{3}$$

- Gravitational potential energy:

$$V = -\frac{GMm}{r} = -\frac{4\pi G m\rho r^2}{3}$$

- Kinetic energy of test particle: $T = \frac{1}{2}m\dot{r}^2$
- Energy conservation: $U = T + V = \frac{1}{2}m\dot{r}^2 - \frac{4\pi G m\rho r^2}{3}$

The Friedmann Equation

- Change to co-moving co-ordinates $\underline{\boldsymbol{x}}$
  - carried along with the expansion
- *a*(t) is the scale factor of the Universe
  - time-dependent magnification factor

$$\underline{\mathbf{r}} = a(t)\underline{\mathbf{x}}$$

$$\dot{x} \overset{def}{=} 0$$

$$U = \frac{1}{2}m\dot{a}^2x^2 - \frac{4\pi G m\rho a^2 x^2}{3}$$

- Defining

$$kc^2 = -\frac{2U}{mx^2}$$

- We find

$$\boxed{\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}}$$

The Friedman equation is incompatible with a static universe with non-zero density, hence the addition of Lambda (Einstein's "Greatest Blunder")

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho + \Lambda}{3} - \frac{kc^2}{a^2}$$