An Excursion in Discrete Geometry

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Agenda

- A gentle introduction to discrete geometry
- Notion of convexity
- Some important theorems
- Conclusion and future directions
- References

What is Discrete Geometry?

- *Discrete geometry* and *combinatorial geometry* study combinatorial properties and constructive methods of discrete geometric objects.
- Questions involve finite or discrete sets of basic geometric objects, such as *points, lines, planes, circles, spheres*, and polygons.
- The subject focuses on the combinatorial properties of these objects, such as how they intersect one another, or how they may be arranged to cover a larger object.
- It is closely related to *combinatorial optimization, computational geometry, digital geometry*, and *geometric graph theory*.
- The subject was mostly developed during the 20th century.

Notion of Convexity

- Let S be a *vector space* or an *affine space* over the real numbers.
- A subset C of S is *convex* if, for all $x, y \in C$ and $\alpha \in [0, 1]$, $\alpha x + (1 - \alpha)y \in C$.
- Equivalently, for all $x, y \in C$, the line segment connecting x and y is entirely included in C.

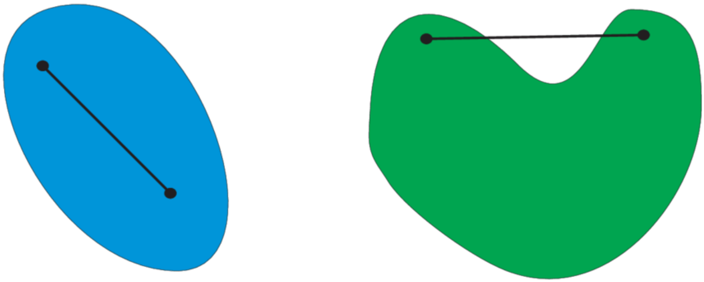

Figure: An example of a convex and a non-convex set.

Convex Hull

- The *convex hull* $CH(X)$ of a set of points $X \in \mathbb{R}^d$ is the *smallest convex set* that contains X.
- It is the intersection of all *convex sets* in $\mathbb{R}^d$ containing X.
- Equivalently, it is the set of all *convex combinations* of points in X.

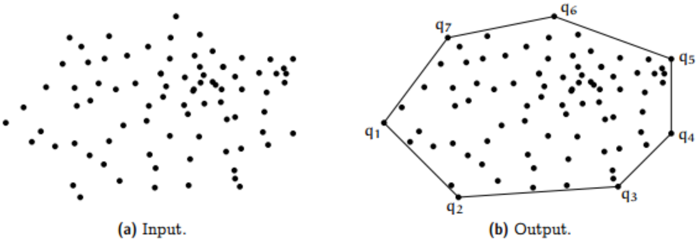

Figure: An example of a convex hull.

Important Theorems in Discrete Geometry

- Caratheodory's Theorem
- Radon's Theorem
- Helly's Theorem
- Minkowski's Theorem
- Centerpoint Theorem
- Ham-Sandwich Theorem
- Sylvester-Gallai Theorem
- Erdos-Szekeres Theorem
- Szemeredi-Trotter Theorem

Caratheodory's Theorem

- For a set $X \in \mathbb{R}^d$, each point $x \in CH(X)$ is a convex combination of at most $d+1$ points of X.
- Equivalently, there is a subset $Y \subseteq X$ consisting of at most $d+1$ points such that $x \in CH(Y)$.

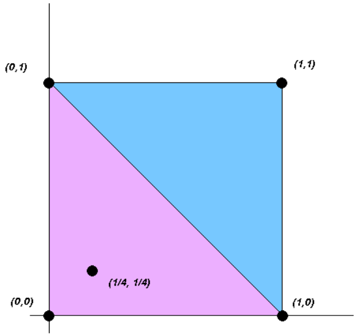

Figure: An illustration of Caratheodory's Theorem.

Radon's Theorem

- Let X be a set of $d + 2$ points in $\mathbb{R}^d$. Then there exist two disjoint subsets $X_1, X_2 \subseteq X$ such that $CH(X_1) \cap CH(X_2) \neq \emptyset$.
- A point $x \in CH(X_1) \cap CH(X_2)$ at the intersection of these convex hulls is called a *Radon point* of the set X.

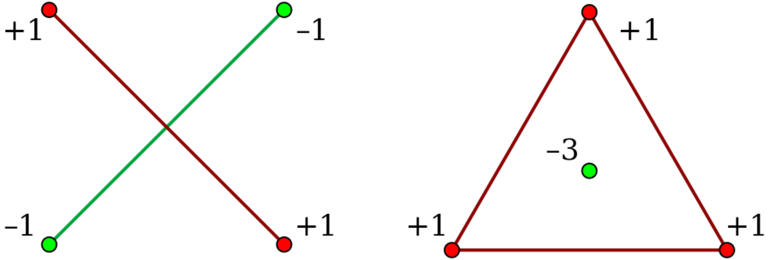

Figure: An illustration of Radon's Theorem.

Helly's Theorem

- Let $C_1, C_2, \ldots, C_n$ be convex sets in $\mathbb{R}^d, n \geq d+1$. Suppose that the intersection of every $d+1$ of these sets is non-empty. Then the intersection of $C_1, \ldots, C_n$ is non-empty, i.e., $\bigcap_{i=1}^{n} C_i \neq \varnothing$.

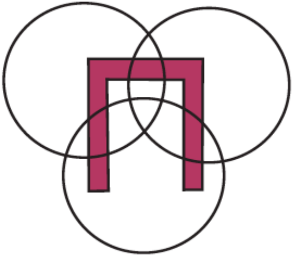

Figure: The importance of convexity in Helly's Theorem.

Minkowski's Theorem

- Suppose that $C \subseteq \mathbb{R}^d$ is a convex, bounded set, symmetric around the origin (i.e., $C = -C$), and $volume(C) > 2^d$. Then C contains at least one *lattice (integer) point* different from 0.

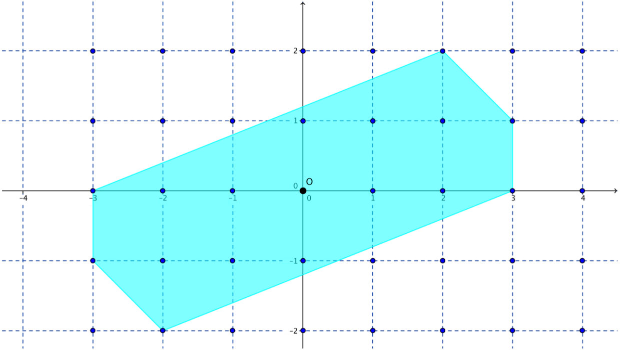

Figure: An illustration of Minkowski's Theorem.

Centerpoint Theorem

- Let P be a set of n points in $\mathbb{R}^d$. There exists a point $c \in \mathbb{R}^d$ such that every closed half-space containing c contains at least $\frac{n}{d+1}$ points of P.
- The point c is called the *centerpoint* of P.
- The notion of centerpoint can be viewed as a generalization of the *median* for one-dimensional data points.

Ham-Sandwich Theorem

- A hyperplane h *bisects* a finite set A having n points, if each of the open half-spaces defined by h contains at most $\lfloor \frac{n}{2} \rfloor$ points of A.
- Every d finite sets in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane.
- In 3-dimensions, the three objects are a chunk of ham and two chunks of bread making a sandwich, all of which can be simultaneously bisected with a single cut (a plane).

- For a finite set of points in the plane, each colored *red* or *blue*, there is a line that simultaneously bisects both the red points and the blue points, i.e., the number of red points on either side of the line is equal and the number of blue points on either side of the line is equal.

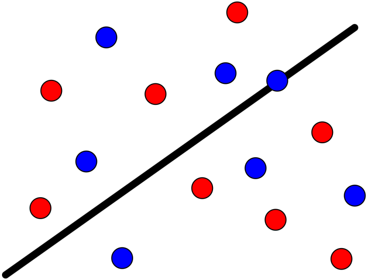

Figure: An illustration of Ham-Sandwich Theorem.

Sylvester-Gallai Theorem

- Let $X \in \mathbb{R}^2$ be a finite set of points, not all on the same line. Then there exists a line containing exactly two points in X.
- Equivalently, if for any two points in X, there exists another point on the line joining these two points, then there is a line containing all the points in X.

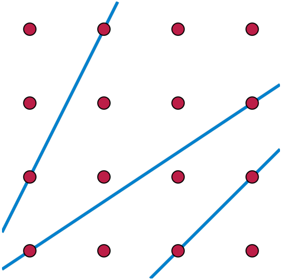

Figure: An illustration of Sylvester-Gallai Theorem.

Erdos-Szekeres Theorem

- A set $X \subseteq \mathbb{R}^d$ is *convex independent* if for every $x \in X$, it is the case that $x \notin CH(X \setminus \{x\})$.
- In the plane, a finite convex independent set is the set of vertices of a convex polygon.
- For every natural number k, there exists a number $n(k)$ such that any $n(k)$-point set $X \subseteq \mathbb{R}^2$ in general position contains a k-point convex independent subset.
- Among any 5 points in the plane in general position (no 3 points are collinear), we can find 4 points forming a convex independent set.

Erdos-Szekeres Theorem continued . . .

- These are examples of *Ramsey-type theorems* – Every sufficiently large structure of a given type contains a *regular* substructure of a prescribed size.

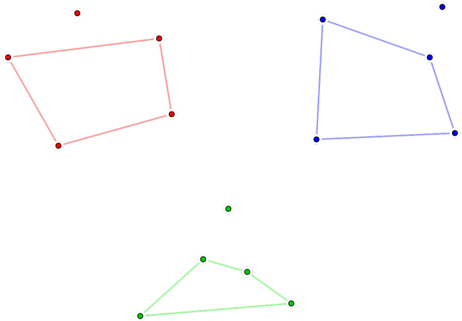

Figure: An illustration of Erdos-Szekeres Theorem.

Szemeredi-Trotter Theorem

- Consider a set P of m points and a set L of n lines in the plane.
- We are interested to find the maximum possible number of their incidences, i.e., pairs (p, ℓ) such that $p \in P, \ell \in L$, and p lies on ℓ.
- Let the number of incidences for specific P and L be $I(P, L)$, and let $I(m, n)$ be the maximum of $I(P, L)$ over all choices of an m-element P and an n-element L.
- A trivial upper bound is $I(m, n) \leq mn$, but it can never be attained unless $m = 1$ or $n = 1$.
- For all $m, n \geq 1$, we have $I(m, n) = O(m^{\frac{2}{3}} n^{\frac{2}{3}} + m + n)$, and this bound is asymptotically tight.

Conclusion and future directions

- Discrete geometry lies at the intersection of *combinatorics, geometry, and graph theory*.
- It borrows ideas and proof techniques from all these areas.
- The algorithms for computing these geometric objects and structures are studied in *computational geometry*.
- There are many open problems in discrete geometry, some of which are given in the references.

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