Towards a General Theory of Closure Invariants in Radio Interferometry

Nithyanandan Thyagarajan
(CSIRO Space & Astronomy/NRAO)

Arxiv
2108.11399
2108.11400

Joseph Samuel
(RRI / ICTS)

Rajaram Nityananda
(APU)

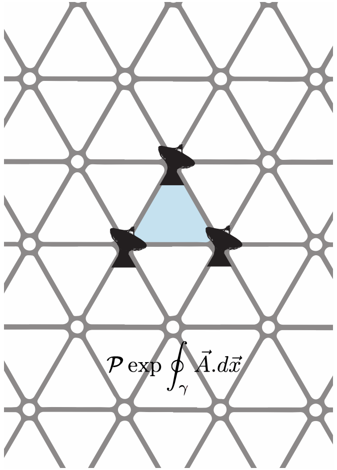


Image credit: Roshni Rebecca Samuel

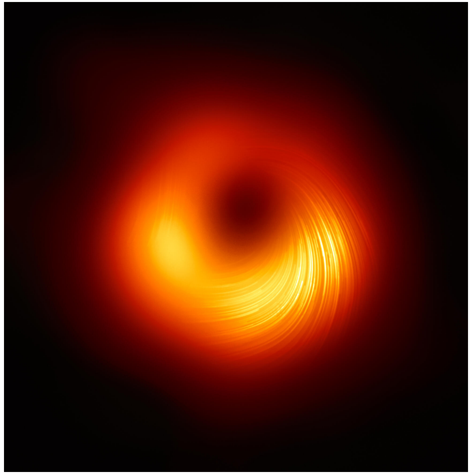
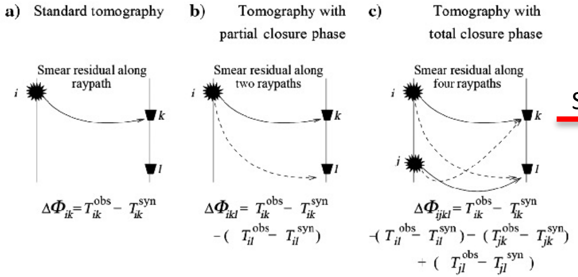


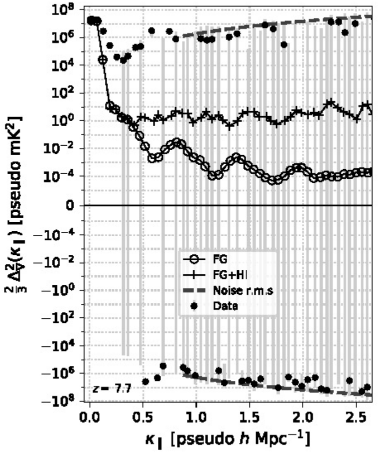
Image credit: EHT collaboration

I acknowledge the Traditional Owners of the land, sea and waters, of the area that we live and work on across Australia. I acknowledge their continuing connection to their culture and pay my respects to their Elders past and present.

Scientific Motivations

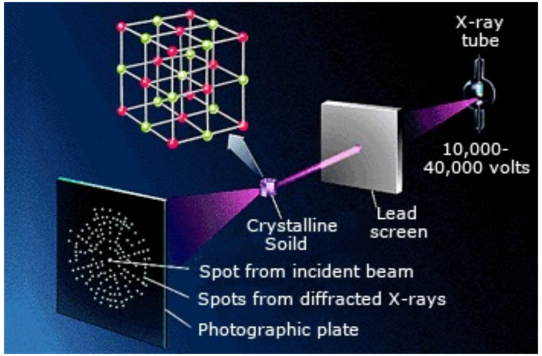


Seismic imaging/ Geophysics, Schuster+2014

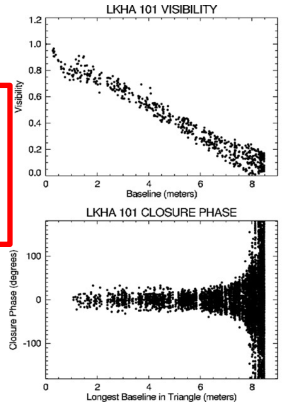


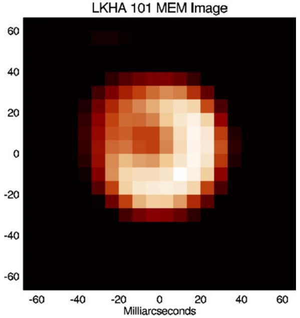
21cm cosmology, Thyagarajan+2020

Invariance to element-based gain corruptions => very valuable in applications where calibration/phase determination is challenging



X-ray crystallography

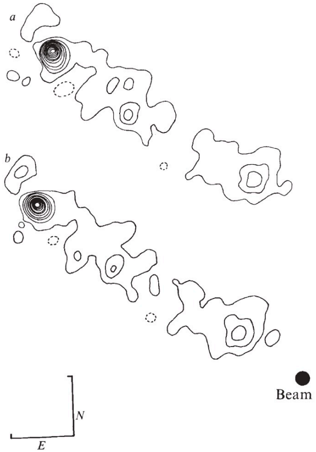




Dust shells around stars
Optical Interferometry
Monnier+2007

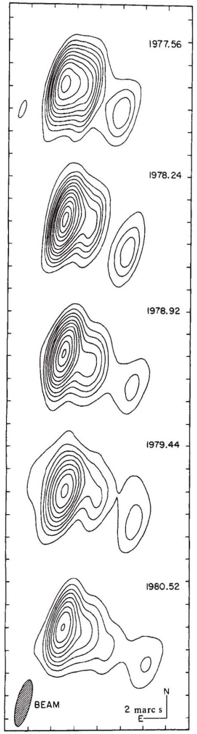
Scientific Motivation: Radio Astronomy

Wilkinson+(1977)



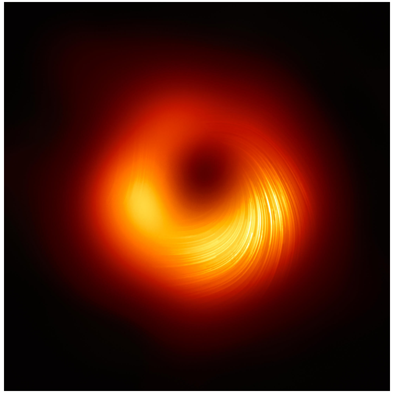
First determination of core-jet morphology of quasar 3C147

Pearson+(1981)



First direct evidence of superluminal expansion (~10c) of relativistic jet in quasar 3C273

Image credit: EHT collaboration



- First μas second scale imaging of event horizon of M87 black hole
- First polarized imaging of magnetic structures inferring the presence of jet launch near the event horizon
- BH parameters verifying GR

Relation to hybrid mapping and self-calibration

Hybrid Mapping

- Minimize χ^2 in closure phases and amplitudes between data and model with a priori information
- Iterative determination of model visibilities/image until convergence

Self-calibration

- Minimize χ^2 in complex visibilities between data and model by solving for antenna-based gains
- Iterative determination of model image through antenna gains until convergence
- Antenna-based => Inherently preserves closure quantities

Wilkinson (1988), TMS (2017)

Limitations of Self-calibration

- Sparse uv-coverage / complex source
- Low S/N
- Deconvolution effects
- Spurious symmetrization
- Local minima and degeneracies, dependence on initial guess, and convergence issues

- "If you can't calibrate, eliminate!" – RN
- Complementary technique using forward-modeling but independent of calibration and deconvolution.
- More robust for low S/N, complex source structure, and/or sparse uv-coverage

Summary of Results

Problems vs Solutions

- Closure phases and amplitudes in co-polar case require different treatments (triangles vs. quadrilaterals)
- Polarimetric "closure traces" (Broderick & Pesce, 2020) was for N=4 requiring auto-correlations
- Prescription unclear for independent and complete set of quadrilateral quantities (when $N \geq 5$) for both co-polar and polarimetric cases

- Unified treatment of all closure quantities using triangular loops as elementary units
- Polarimetric closure invariants solved using ideas from lattice gauge theories and Lorentz invariance properties
- Prescription provided for complete and independent set for arbitrarily large N for co-polar and polarimetric cases, along with numerical verification

Numerology

Co-polar measurements

- Correlations: N(N-1)/2 complex numbers => N(N-1) real measurables
- Gains: N complex numbers => 2N real gain parameters
- Only phase differences matter => N-1 unknown phases in gains (after fixing one reference phase)
- N amplitude parameters unknown
- The rest $\underline{N^2\text{-}3N+1^*}$ should be invariant

* 1 more from auto-correlation

Polarimetric measurements

- Correlations: N(N-1)/2 complex 2x2 matrices => 4N(N-1) real measurables
- Gains: N complex 2x2 matrices => 8N real gain parameters
- After fixing one reference phase, 8N-1 unknown real parameters in gains
- The rest $\underline{4N^2\text{-}12N+1^{**}}$ should be invariant

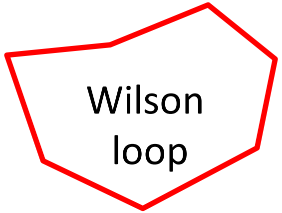
** N=3 is special

** 4 more from auto-correlation

Theory of Closure Invariants

Motivation for new form

- "Flux" in a Closed loop (Closure)
- Product cancels phase not amplitudes
- Ratios cancel amplitudes but not phases
- Cancel post-factor of last term with pre-factor of next term => hat operation
- Even number terms required (covariants)
- Taking trace will eliminate all gains (Broderick & Pesce, 2020)
- Even terms (≥ 4) => Prescription unclear for independent set (~N^2 out of ~N^4 combinations)



Wilson (1974)

$$\mathcal{P}e^{i\oint_C A_\mu dx^\mu} \to g(x)\mathcal{P}e^{i\oint_C A_\mu dx^\mu} g^{-1}(x)$$

$$W_C := \mathrm{Tr}\,(\,\mathcal{P}\exp i\oint_C A_\mu dx^\mu\,)\,.$$

$$\mathcal{B}_{abc} = C_{ab}C_{bc}C_{ca}$$ ✗

$$\mathcal{Q}_\Gamma = C_{ab}\,C_{bc}^{-1}\,C_{cd}\,C_{da}^{-1}$$ ✗

$$\widehat{\mathbf{P}} = (\mathbf{P}^{-1})^\dagger = (\mathbf{P}^\dagger)^{-1}$$

$$\mathcal{C}_\Gamma = C_{ab}\widehat{C}_{bc}C_{cd}\widehat{C}_{da}$$ ✓

$$\boldsymbol{\mathcal{C}} \overset{\mathcal{G}}{\longmapsto} \mathbf{G}_0\boldsymbol{\mathcal{C}}\mathbf{G}_0^{-1}$$

Fundamental Triangular Plaquettes

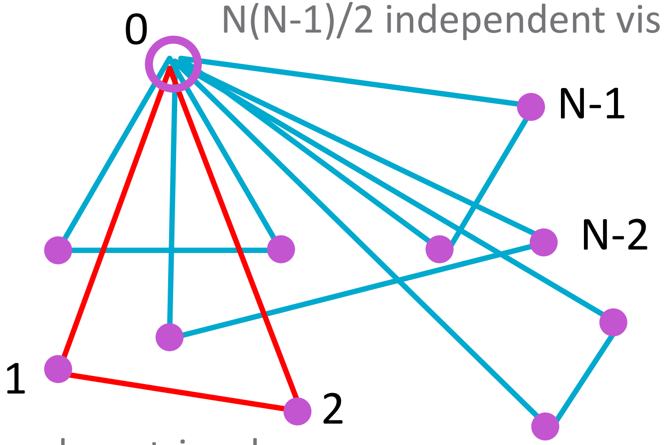
- Well-known prescription for independent triangles
- Independent triangles obtained by pinning one vertex at 0 and varying the other two vertices => (0,a,b)
- "Advariant": closed loop, odd terms (≥ 3) with alternate terms hatted

On (0,a,b)

$$\boldsymbol{A}_{ab} := \mathrm{C}_{0a}\widehat{\mathrm{C}}_{ab}\mathrm{C}_{b0}$$

$$\boldsymbol{A} \overset{\mathcal{G}}{\mapsto} \mathrm{G}_0 \boldsymbol{A} \mathrm{G}_0^{\dagger}$$

N-element array
N(N-1)/2 independent visibilities



base triangle (0,1,2) $\mathcal{A}_{\Delta 1}$

N_{Δ} =(N-1)(N-2)/2 independent triangles (pinned at a vertex 0) and advariants, $\mathcal{A}_{\Delta k}$

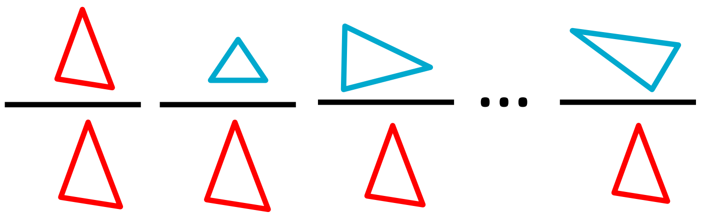
Co-polar invariants

Method 1: Ratios between advariants

$$\mathcal{A} \overset{\mathcal{G}}{\longmapsto} \mathrm{G}_0 \mathcal{A} \mathrm{G}_0^\dagger$$

Abelian (commutative) property => $\mathcal{A}_{ab} \overset{\mathcal{G}}{\longmapsto} |G_0|^2 \mathcal{A}_{ab}$

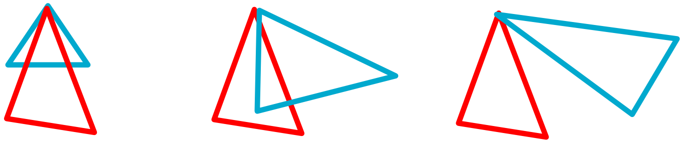
- All (N-1)(N-2)/2 complex advariants have one unknown scale factor $|G_0|^2$
- Advariants are not invariants
- (N-1)(N-2) real numbers in advariants
- Take ratio to eliminate scale factor



N^2 - 3N + 1 real invariants (complete + independent)

Co-polar invariants
Method 2: Pair triangles / advariants

Covariants = pair advariants with base advariant



$$\mathcal{C}_{\Delta_1;\Delta_\ell} = \mathcal{A}_{\Delta_1}\widehat{\mathcal{A}}_{\Delta_\ell} \qquad \mathcal{C} \overset{\mathcal{G}}{\longmapsto} \mathbf{G}_0\mathcal{C}\mathbf{G}_0^{-1}$$

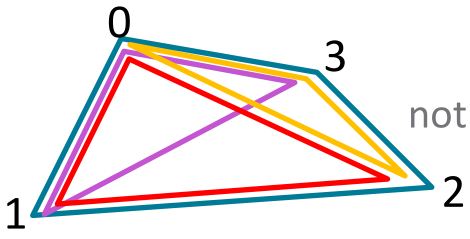
- Covariants are invariants because GL(1,C) matrices commute (Abelian)
- N_Δ pairs (=2 N_Δ) real numbers but Δ_1 paired with itself has trivial unit amplitude

2 N_Δ-1 = N^2 - 3N + 1 real invariants

Co-polar invariants
Closure Phases + Closure Amplitudes

$$\mathcal{A}_{ab} := \mathrm{C}_{0a}\widehat{\mathrm{C}}_{ab}\mathrm{C}_{b0}$$

- Consider N=4 (for example)
- 3 independent triangles => (0,1,2), (0,1,3), (0,2,3)
- Advariant paired with itself has unit amplitude and only phase
- Other pairs contain amplitude and phase invariants
- Consistent with standard theory:
  - 3 closure phases from 3 independent triangles
  - 2 independent closure amplitudes (3 in total - 1 dependent)



(1,2,3) is not independent

$$I_{\Delta_1;\Delta_1} = \mathcal{A}_{\Delta_1}\widehat{\mathcal{A}}_{\Delta_1} = e^{2i\phi_{\Delta_1}}$$

$$I_{\Delta_1;\Delta_2} = \mathcal{A}_{\Delta_1}\widehat{\mathcal{A}}_{\Delta_2} = \frac{|C_{01}||C_{23}|}{|C_{21}||C_{03}|}e^{i(\phi_{\Delta_1}+\phi_{\Delta_2})}$$

$$I_{\Delta_1;\Delta_3} = \mathcal{A}_{\Delta_1}\widehat{\mathcal{A}}_{\Delta_3} = \frac{|C_{20}||C_{31}|}{|C_{21}||C_{30}|}e^{i(\phi_{\Delta_1}+\phi_{\Delta_3})}$$

More on Covariants

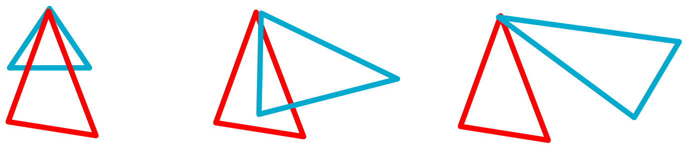
Pairing even advariants (min. 2) with alternate terms hatted is a covariant

In general, covariants contain 6 terms (traversing two triangles)

$$|C_{01}||C_{21}|^{-1}|C_{20}||C_{a0}|^{-1}|C_{ab}||C_{0b}|^{-1}$$

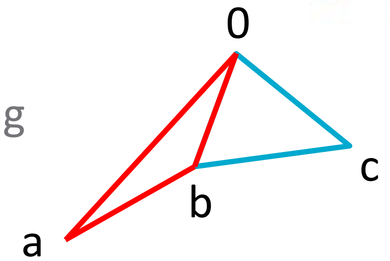
When triangles share edges, covariants contain 4 terms due to edge terms cancelling

$$|C_{01}||C_{21}|^{-1}|C_{2b}||C_{0b}|^{-1}$$



$$\mathcal{C}_{\Delta_1;\Delta_\ell} = \underline{\mathcal{A}_{\Delta_1}\widehat{\mathcal{A}}_{\Delta_\ell}}$$

$$\mathcal{C} \overset{\mathcal{G}}{\longmapsto} \mathrm{G}_0\mathcal{C}\mathrm{G}_0^{-1}$$



$$\mathcal{C}_{\Gamma_0} = \underline{\mathcal{A}_{ab}\widehat{\mathcal{A}}_{bc}}$$

Properties of Advariants

Co-polar Interferometry

- Visibilities, gains, and advariants are complex scalars => GL(1,C) and Abelian
- Advariants scale as $|G_0|^2$

$$\mathcal{A}_{ab} \overset{\mathcal{G}}{\mapsto} |G_0|^2 \mathcal{A}_{ab}$$

Polarimetric Interferometry

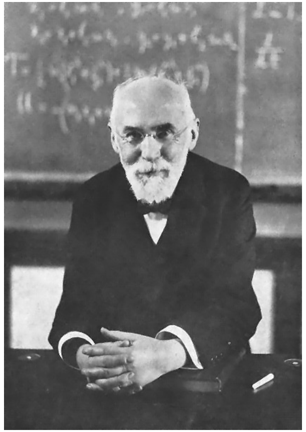
- Visibilities, gains, and advariants are general 2x2 complex matrices => GL(2,C) and non-Abelian

$$\boldsymbol{\mathcal{A}} \overset{\mathcal{G}}{\mapsto} \mathrm{G}_0 \boldsymbol{\mathcal{A}} \mathrm{G}_0^\dagger$$

- Express advariants as co-efficients of Pauli matrices $\boldsymbol{\mathcal{A}} = z^\mu \boldsymbol{\sigma}_\mu$
- Advariants scale as $|G_0|^2$ +undergo Lorentz transformations

$$\boldsymbol{z} \overset{\mathcal{G}}{\mapsto} |\det(\mathbf{G}_0)|^2 \boldsymbol{\Lambda} \boldsymbol{z}$$

Borrowing from Lorentz transformations and Invariance



Hendrik Lorentz
(1853 – 1928)

Lengths of 4-vectors in space-time are invariant under Lorentz transformations

$$(\Delta s)^2 = (\Delta ct)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2.$$

$$M = \begin{bmatrix} I + Q & U - iV \\ U + iV & I - Q \end{bmatrix}$$

Stokes parameters are a Lorentz 4-vector

$$I^2 - Q^2 - U^2 - V^2 = (I'^2 - Q'^2 - U'^2 - V'^2)\, det(G_0 G_0^*)$$

M. C. Britton, Radio Astronomical Polarimetry and the Lorentz Group, ApJ (2000), 532, 1240

(for a single antenna system)

Express advariants as co-efficients of Pauli matrices

$$\boldsymbol{\mathcal{A}} = z^{\mu} \boldsymbol{\sigma}_{\mu}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

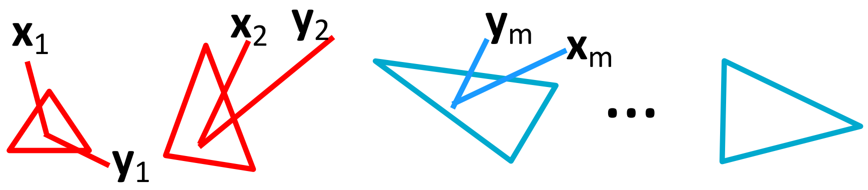
Advariants scale as $|G_0|^2$ +undergo Lorentz transformations

$$\boldsymbol{\mathcal{A}} \overset{\mathcal{G}}{\mapsto} \mathbf{G}_0 \boldsymbol{\mathcal{A}} \mathbf{G}_0^{\dagger} \qquad \boldsymbol{z} \overset{\mathcal{G}}{\mapsto} |\det(\mathbf{G}_0)|^2 \, \boldsymbol{\Lambda} \boldsymbol{z}$$

Inner products between 4-vectors are invariants

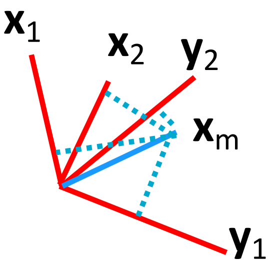
Closure Invariants in Polarimetric Interferometry

- Each advariant split into two real 4-vectors (**x** and **y**), both transform as Lorentz 4-vectors
- Use four 4-vectors from two base triangles to construct a frame
- Four basis 4-vectors yield 10 inner products
- Pairs of 4-vectors from the rest of the triangles are projected on the basis vectors => $4\,(2\,N_{\Delta}-4)$ inner products
- Total of $10 + 4\,(2\,N_{\Delta}-4) = 8N^2-12N+2$ inner products, all containing an unknown scale factor $|G_0|^2$ (eliminated by taking ratios)
- => $8N^2-12N+1$* real invariants



Two base triangles Δ_1 and Δ_2

N_{Δ}-2 remaining triangles



$8N^2-12N+1$* real invariants

* N=3 is special

* 4 more from auto-correlation

Connection to recent work (Closure Traces)

$$\mathcal{C}_{\Gamma} = C_{ab}\widehat{C}_{bc}C_{cd}\widehat{C}_{da}$$

$$\mathcal{C} \overset{\mathcal{G}}{\mapsto} \mathrm{G}_0\mathcal{C}\mathrm{G}_0^{-1}$$

- Tr(ABC) = Tr(CAB)
- Traces on even loops (covariants) eliminates all gains (Broderick & Pesce, 2020)
- Required using even loops and auto-correlations
- Restricted to N=4
- Minor discrepancy (independent but not a complete set)

$$\mathcal{C}_{\Gamma_0} = \boldsymbol{\mathcal{A}}_{ab}\widehat{\boldsymbol{\mathcal{A}}}_{bc}$$

Tr(covariants) = ratio of inner products

$$\mathcal{T}_{\Delta_1\Delta_2} = \frac{1}{2}\,\mathrm{tr}[\boldsymbol{\mathcal{A}}_{12}\widehat{\boldsymbol{\mathcal{A}}}_{23}] = \frac{\mathbf{z}_{12}\cdot\mathbf{z}_{23}^*}{(\mathbf{z}_{23}\cdot\mathbf{z}_{23})^*}$$

- Formalism is easier when using inner products
- Prescription to obtain complete and independent set easily generalisable and scalable to any value of N

Numerical Verification

Method 1: Correlations vs. Gains

$$[\delta C_{ab}] = \left[\frac{\partial C_{ab}}{\partial G_a}\right][\delta G_a]$$

$[\delta C_{ab}]$: 4N(N-1); Jacobian: 4N(N-1) rows, 8N columns; $[\delta G_a]$: 8N

Jacobian 4N(N-1) x 8N

Rank of Jacobian via SVD = 8N-1

Method 2: Invariants vs. Correlations

$$[\delta I] = \left[\frac{\partial I}{\partial S_{ab}}\right][\delta S_{ab}]$$

$[\delta I]$: N_{inv}; Jacobian: N_{inv} rows, 4N(N-1) columns; $[\delta S_{ab}]$: 4N(N-1)

Jacobian

N_{inv} x 4N(N-1)

Rank of Jacobian via SVD = $4N^2-12N+1$*

* N=3 is special

* 4 more from auto-correlation

Numerology and formalism verified numerically

Applications & Future work

- Immediate application:
  - Forward modeling and constraining of BH models using EHT data and EHT-like simulations using only closure quantities
  - Identify correspondence between closure quantities and BH parameters

- Future work:
  - Explore other astronomical applications, including 21cm cosmology using interferometer arrays
  - Explore applications outside astronomy
  - Identify potentially more correspondences in other branches of physics and develop theory

Summary

- Ideas borrowed from lattice gauge theories and Lorentz transformation properties
- Unified treatment of closure invariants for co-polar and polarimetric interferometry using only elementary triangular plaquettes
- Familiar closure phases and amplitudes naturally emerge
- Prescription for determining a complete and independent set of interferometric invariants for arbitrary array sizes
- Verified numerically
- Applications: EHT Blackhole modeling, 21cm cosmology, …