Fundamental Radio Interferometry
II. Extensions and Geometry

Rick Perley, NRAO/Socorro

CSIRO Radio School, Narrabri, NSW

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Topics – Part I

- This lecture contains two parts: Extensions to the basic theory, and geometrical considerations.
- This part extends the basic theory to include various practical details.
- We will cover:
  - Real Sensors (aka ‘antennas’)
  - Finite bandwidth
  - Rotating reference frames (source motion)
  - Finite time averaging
  - Local Oscillators and Frequency Downconversion
- But I won’t discuss polarization, sensitivity or calibration. These are topics for following lectures.

Review

- In the previous lecture, I set down the principles of Fourier synthesis imaging.
- I showed: $$V_\upsilon(\mathbf{b}) = R_C - iR_S = \iint I_\nu(\mathbf{s}) e^{-2\pi i \nu \mathbf{b}\cdot\mathbf{s}/c} d\Omega$$

  where the intensity $I_\nu(\mathbf{s})$ is a real function, and the visibility V(**b**) is complex and Hermitian (V(**b**) = V*(-**b**))
- The model used for the derivation was idealistic – not met in practice. We presumed:
  - Isotropic Sensors (no dependence on direction)
  - Monochromatic radiation
  - Stationary reference frame (no motion)
  - No frequency conversions
- I now relax, in turn, these restrictions.

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Real Sensors

- The idealized isotropic sensors assumed earlier don't exist.
- Real sensors (antennas) have a directional voltage gain pattern A($\mathbf{s}$, $\mathbf{s}_0$), where $\mathbf{s}$ is a general direction, and $\mathbf{s}_0$ is the antenna pointing direction.
- The gain pattern is easily incorporated into the formalism, once we realize that it attenuates the actual sky brightness. We can write:

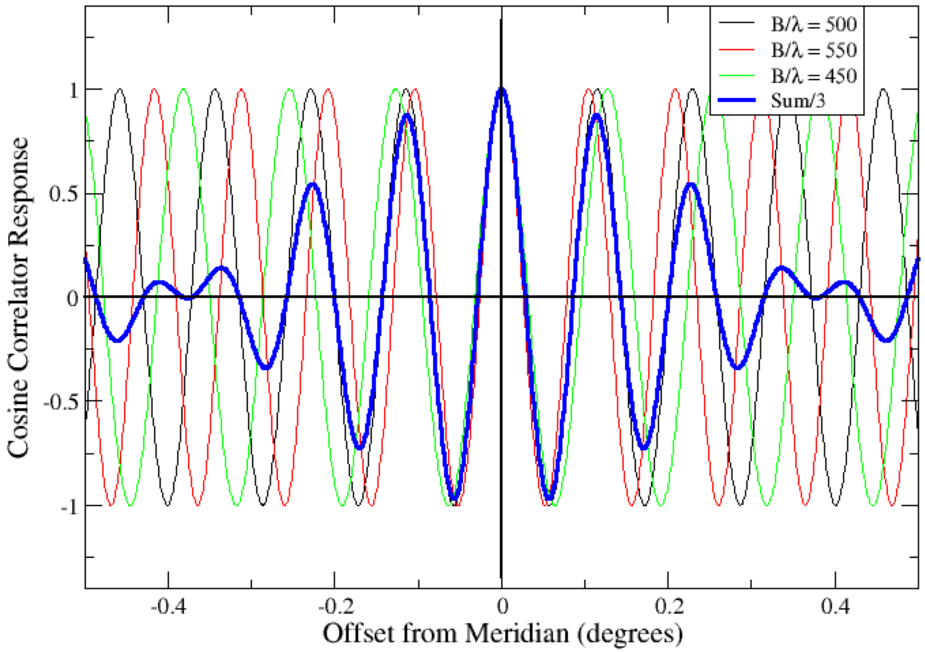
$$V_\upsilon(\mathbf{s}_0,\mathbf{b}) = \iint A_1(\mathbf{s},\mathbf{s_0}) A_2^*(\mathbf{s},\mathbf{s_0}) I_\nu(s) e^{-2\pi i \nu \mathbf{b}\cdot\mathbf{s}/c} d\Omega$$

- Here, A_1 and A_2 are the complex voltage gain functions for the two antennas. (Complex means it contains both amplitude and phase).
- Note that if the antenna patterns are identical, and pointing in the same direction, then $A_1 A_2^* = A^2$, the power gain response.
- The effect on the output image is an attenuated (modified) representation of the true sky: $I_{\text{mod}} = A^2 I$
- We can correct for this in the image by dividing by the beam inverse.
  - This is valid only when the beams have azimuthal symmetry!

Effect of Finite Bandwidth

- Real interferometers must have a finite, non-zero, bandwidth.
- Each frequency component generates a fringe pattern with angular separation of λ/B. (B = the baseline length – a fixed number).
- All fringe patterns have a maximum at the n=0 fringe (meridional plane).
- The patterns get increasingly out of step as n gets larger.

- A simple illustration – three wavelength components from the same physical baseline.
- Baselines of u=450, 500, and 550 wavelengths.
- The net result (thick blue line) is the (normalized) sum over all components.

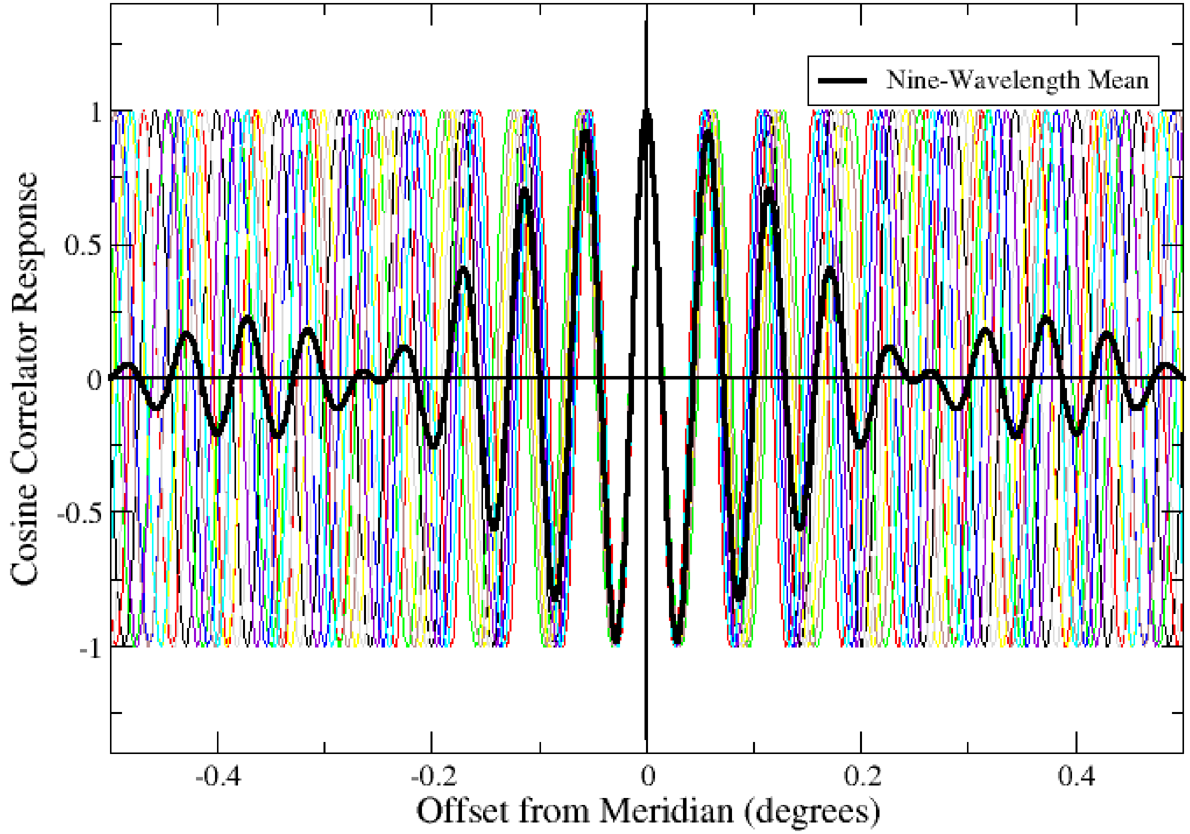


Bandwidth Effect (cont.)

- With more components (nine equal ones in this case), the summed response (thick black lilne) begins to look like a 'wave packet'.
- This shows the 'COS' component – the SIN component is shifted so the null is at the origin.

The black trace is what we actually observe.

Note that I added the nine equal components to make this response. In fact, the bandpass is not flat, and has phase and amplitude variations – the actual sum is complex.



The Effect of Bandwidth -- Analysis.

- To find the finite-bandwidth response, we integrate our fundamental response over a frequency response G(ν), of width Δν, centered at ν_0

$$V = \iint \left(\frac{1}{\Delta \nu} \int_{\upsilon_0 - \Delta\upsilon/2}^{\upsilon_0 + \Delta\upsilon/2} I(\mathbf{s},\upsilon) G_1(\upsilon) G_2^*(\upsilon) e^{-i2\pi\upsilon\tau_g} d\upsilon \right) d\Omega$$

- If the source intensity does not vary over the bandwidth, and the gain parameters G_1 and G_2 are square (= 1.0) and real, then

$$V = \iint I_\upsilon(\mathbf{s}) \frac{\sin(\pi\tau_g\Delta\upsilon)}{\pi\tau_g\Delta\upsilon} e^{-2i\pi\nu_0\tau_g} d\Omega = \iint I_\upsilon(\mathbf{s}) \mathrm{sinc}(\tau_g\Delta\upsilon) e^{-2i\pi\nu_0\tau_g} d\Omega$$

where the **fringe attenuation function**, sinc(x), is defined as:

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

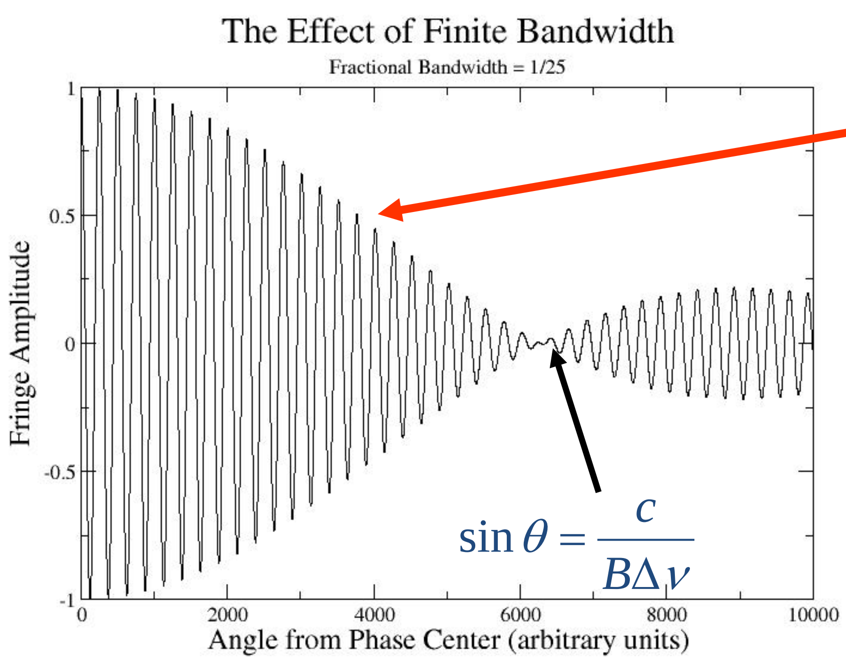
The attenuation function has nulls at x = +/- n.

Bandwidth Effect Example

- For a square bandpass, the bandwidth attenuation reaches a null when $\tau_g \Delta\nu$ = 1, or

$$\sin\theta = \frac{\lambda}{B}\frac{\upsilon_0}{\Delta\upsilon} = \frac{c}{B\Delta\nu}$$

- For the Jansky VLA, $\Delta\nu$ = 1 MHz, and B = 35 km, (A configuration) then the null occurs at about 30 arcminutes off the meridian.
- A 10% loss in fringe amplitude occurs at an offset of ~7.5 arcminutes.
- This is a real loss of visibility amplitude – because the effect depends on offset angle and baseline length, it effectively limits the resolution.



Fringe Attenuation function:

$$\mathrm{sinc}(\tau_g \Delta\upsilon) = \mathrm{sinc}\left(\frac{B\Delta\upsilon}{c}\sin\theta\right)$$

Number of fringes between peak and null:

$$N \sim \frac{c}{B\Delta\upsilon}\frac{B}{\lambda} \sim \frac{\upsilon}{\Delta\upsilon}$$

Observations off the Meridian

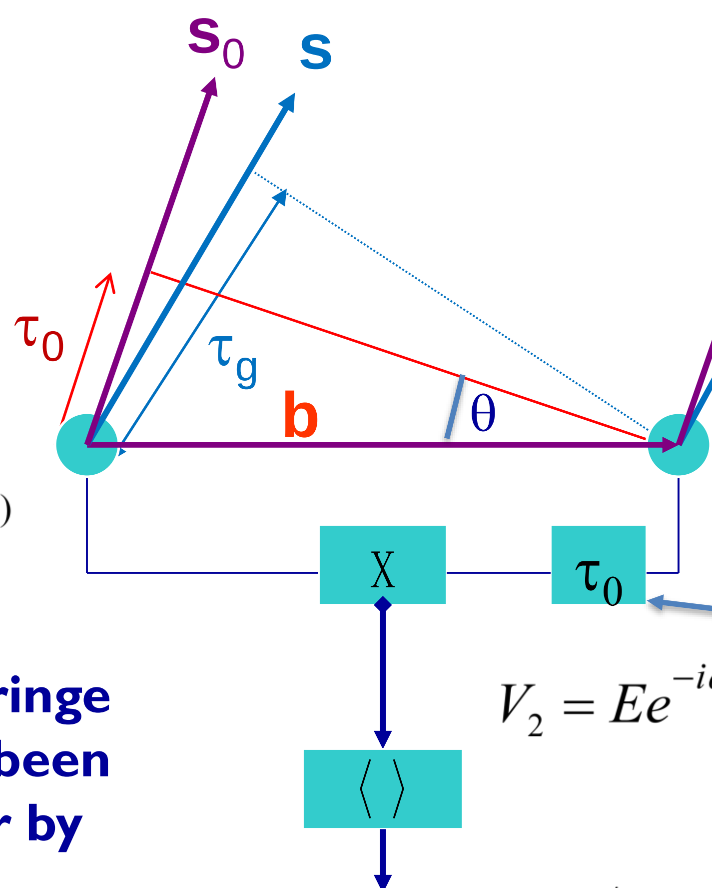
- In our basic scenario -- stationary source, stationary interferometer -- the effect of finite bandwidth will strongly attenuate the fringe amplitudes from sources not on the meridional plane.
- Since each baseline has its own meridional plane, the only point on the sky free of attenuation for all baselines is a small angle around the zenith.
- Hence, for our model (fixed) interferometer, we can only observe objects within a few arcminutes of the central fringe.
- Suppose we wish to observe an object far from the meridional plane? What to do?
  - Reducing the bandwidth is not the answer! (1 kHz needed …)
- Best way is to shift the entire 'fringe pattern' to the position of interest by adding time delay to the signal coming from the antenna closer to the source.

Adding Time Delay

Two delays:

$$\tau_g = \mathbf{b}\cdot\mathbf{s}/c$$

$$\tau_0 = \mathbf{b}\cdot\mathbf{s}_0/c$$



$$V_1 = Ee^{-i\omega(t-\tau_g)}$$

$$V_2 = Ee^{-i\omega t}$$

$$V_2 = Ee^{-i\omega(t-\tau_0)}$$

$\mathbf{s}_0$ = reference (delay) direction

$\mathbf{s}$ = general direction vector

New box: Time delay

The entire fringe pattern has been shifted over by angle

$\sin\theta = c\tau_0/b$

$$V = \langle V_1 V_2^* \rangle = E^2 e^{-i[\omega(\tau_0-\tau_g)]}$$

$$= E^2 e^{i2\pi[\nu\mathbf{b}\cdot(\mathbf{s}-\mathbf{s}_0)/c]}$$

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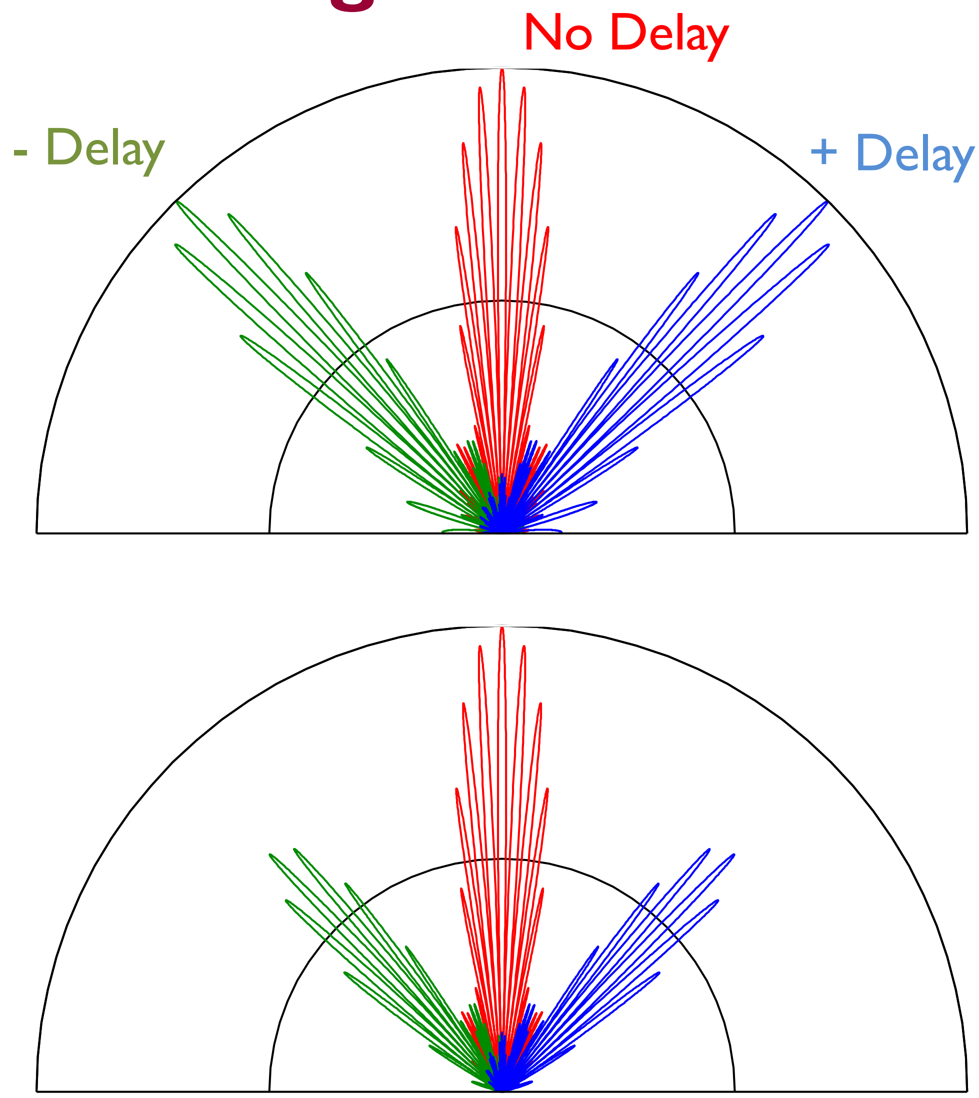
Illustrating Delay Tracking

- Top Panel:

  Delay has been added and subtracted to move the delay pattern maximum to the source location, presuming an isotropic sensor.

- Bottom Panel:

  A cosinusoidal sensor pattern is added, to illustrate losses from a fixed sensor.



Observations from a Rotating Platform

- Most interferometers are built on the surface of the earth – a rotating platform. From the observer's perspective, sources move across the sky.
- Since we know how to adjust the interferometer delay to move its coherence pattern to the direction of interest, it is a simple step to continuously move the pattern to follow a moving source.
- All that is necessary is to continuously add time delay, with an accuracy $\delta\tau \sim 1/(10\Delta\nu)$ to minimize bandwidth loss.
- But there's one more issue to keep in mind, which is that of phase. The source is moving through the fringe pattern, which can be quite rapid.

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Phase Tracking …

- Adding time delay will prevent bandwidth losses for observations off the baseline's meridian.
- Delay insertion is finite – not continuously variable.
- Between delay settings, the source is moving through the interferometer pattern – a rapidly changing phase.
- The 'natural fringe rate' – due to earth's rotation, is given by

$$\upsilon_f = u\omega_e \cos\delta \quad \text{Hz}$$

- Where u = B/λ, the (E-W) baseline in wavelengths, and ω_e =7.3x10^{-5} rad/s is the angular rotation rate of the earth.
- For a million-wavelength baseline, ν_f ~ 70 Hz – that's fast.
- If we leave things this way, we have to sample the output at least twice this rate. A lot of data!

Following a Moving Object.

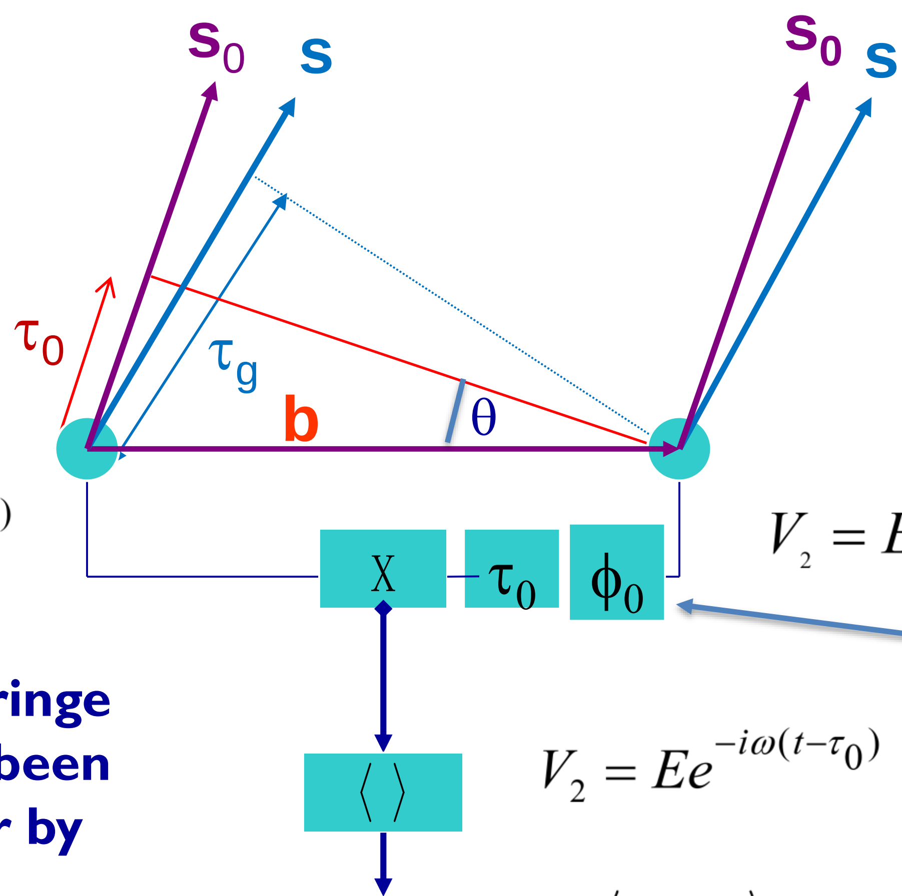
- There is **no useful information** in this fringe rate – it's simply a manifestation of the platform rotation.
- Tracking, or 'stopping' the fringes greatly slows down the post-correlation data processing/archiving needs.
- To 'stop' the fringes, we must shift the signal phases much quicker than the rate of delay adjustments.
- How fast? For a 1 million-wavelength baseline (VLA A-configuration, X-band, 2 MHz channelwidth)
  - Tracking delay: $$\upsilon_d >> \frac{\Delta\upsilon}{\upsilon}\frac{B}{\lambda}\omega_e \cos\delta \sim 0.02\,\mathrm{Hz}$$
  - Tracking fringe: $$\upsilon_f >> \frac{B}{\lambda}\omega_e \cos\delta \sim 70\ \mathrm{Hz}$$

Adding Phase Shift

Two times:

$$\tau_g = \mathbf{b} \cdot \mathbf{s} / c$$

$$\tau_0 = \mathbf{b} \cdot \mathbf{s}_0 / c$$



$$V_1 = Ee^{-i\omega(t-\tau_g)}$$

$$V_2 = Ee^{-i\omega t}$$

$$V_2 = Ee^{-i\omega(t-\tau_0)}$$

$\mathbf{s}_0$ = reference (delay) direction

$\mathbf{s}$ = general direction vector

New box: Phase Shift

The entire fringe pattern has been shifted over by angle

$\sin\theta = c\tau_0/b$

$$V = \langle V_1 V_2^* \rangle = E^2 e^{-i[\omega(\tau_0 - \tau_g)]}$$

$$= E^2 e^{i2\pi[\upsilon \mathbf{b} \cdot (\mathbf{s} - \mathbf{s}_0)/c]}$$

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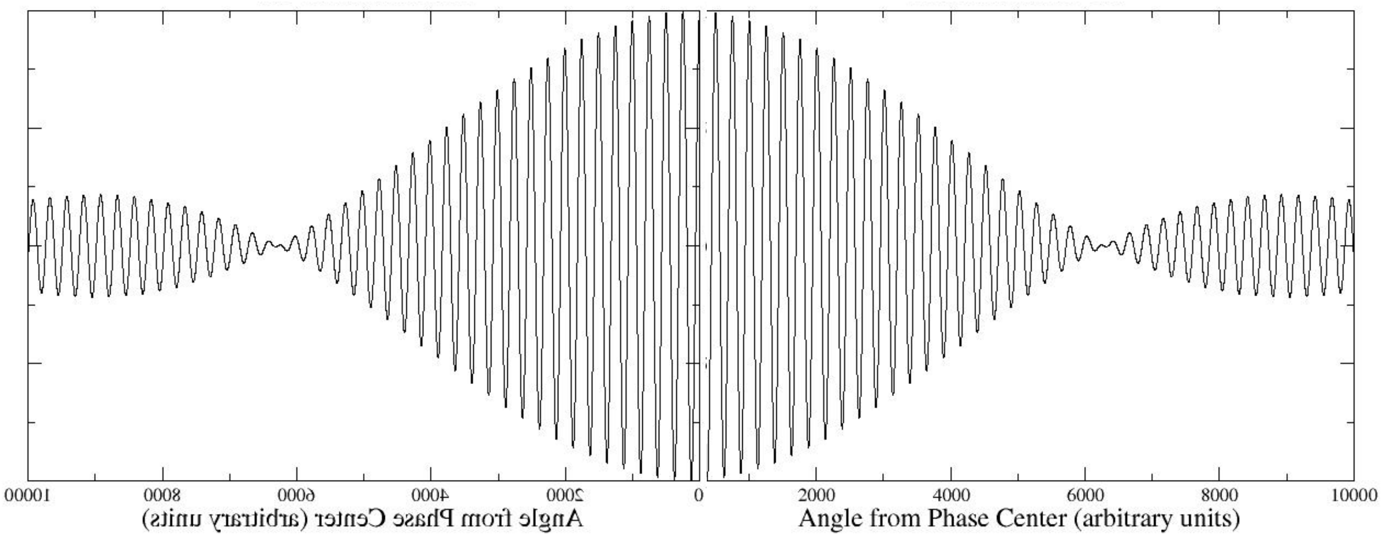
Emphasis:

- Shown again is the fringe pattern of a real wide-band baseline.
- To avoid delay losses, we **must** re-set delays before the source moves too far down the delay pattern.
- To avoid ridiculous sample rates, we need to also shift the signal phase.
- These operations done in the correlator

Source moves this way →



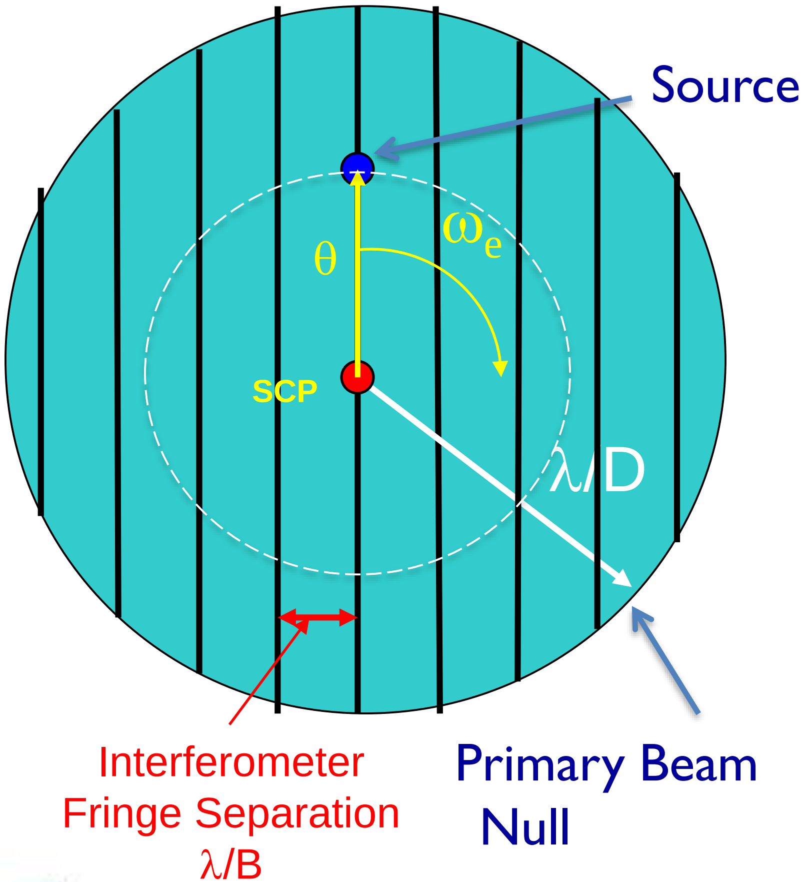
Tracking the phase requires much faster adjustments than maintaining amplitude. The process is much easier if the phase adjustment is done independently of delay (which it is).

Differential Time Averaging Loss

- We can track a moving source, continuously adjusting the delay and phase to move the fringe pattern with the source.
- This does two good things:
  - Slows down the data recording needs
  - Prevents bandwidth delay losses.
- But, you cannot increase the time averaging indefinitely.
- The reason is that the fringe tracking mechanism is correct for only one point in the sky. All others have a different rate.
- So while you can reduce the observed fringe rate to zero for any given place, all other directions retain a differential rate.
- The limit to time averaging set by this differential.

Time-Averaging Loss Timescale

Simple derivation of fringe period, from observation at the SCP.



- Turquoise area is antenna primary beam on the sky – diameter = λ/D
- Interferometer coherence pattern has spacing = λ/B
- Sources in sky rotate about SCP at angular rate:

  $\omega_e = 7.3 \times 10^{-5}$ rad/sec.

- Minimum time taken for a source to move by λ/B at angular distance θ is:

$$t = \frac{\lambda}{B} \frac{1}{\omega_E \theta}$$

- For a source at the primary beam, $\theta = \lambda/D$, so at that radius:

$$t = \frac{D}{B} \frac{1}{\omega_E}$$

- For the VLA in A configuration, t~10 seconds.

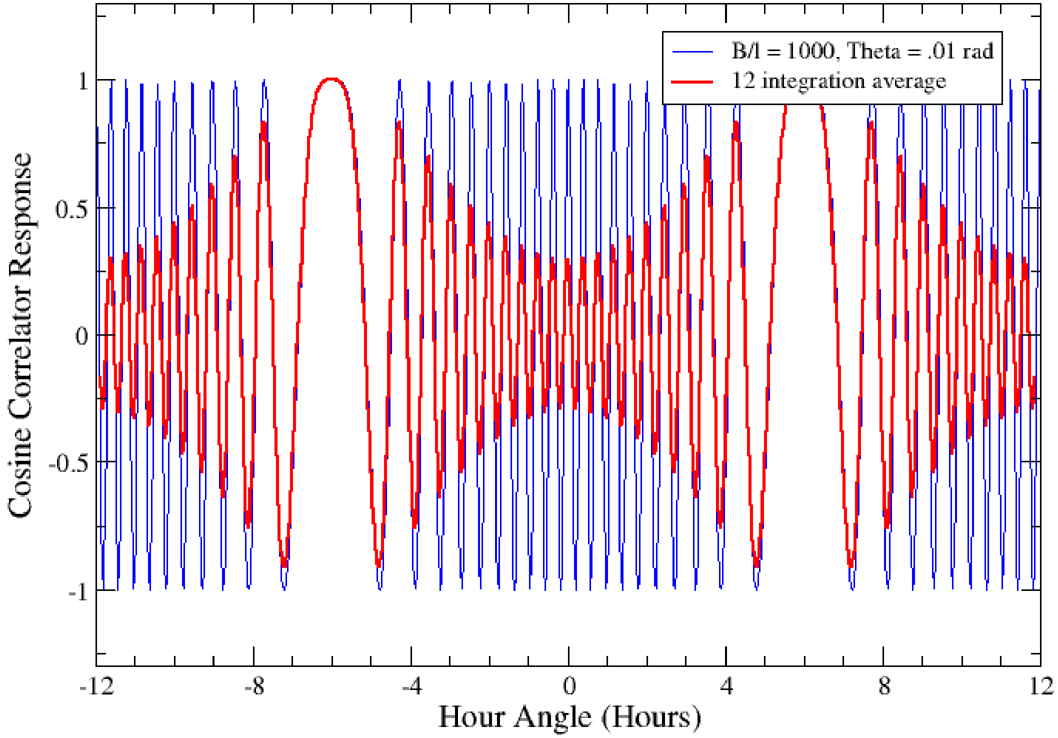
Illustrating Time Averaging Loss

- An object located away from the fringe tracking center moves through the fringe pattern as the earth rotates.
- It makes one cycle around in 24 hours.
- If we average the correlation products for too long a period, a loss in fringe amplitude will result.

Illustrating time average loss.

Blue trace: the fringe amplitude with no averaging.

Red trace: Amplitude after averaging for 12 'samples'.



Time-Averaging Loss

- So, what kind of time-scales are we talking about now?
- How long can you integrate before the differential motion destroys the fringe amplitude?
- **Case A:** A 25-meter parabaloid, and 35-km baseline, for source at primary beam null:
  - $t = D/(B\omega_e) = 10$ seconds. (independent of observing frequency).
- **Case B**: Whole Hemisphere for a 35-km baseline:
  - $t = \lambda/(B\omega_e)$ sec = 83 msec at 21 cm.
- Averaging for durations comparable to these will cause severe attenuation of the visibility amplitudes.
- To prevent 'delay losses', your averaging time must be much less than this.
  - Averaging time 1/10 of this value normally sufficient to prevent time loss.

The Heterodyne Interferometer: LOs, IFs, and Downcoversion

- This would be the end of the story (so far as the fundamentals are concerned) if all the internal electronics of an interferometer would work at the observing frequency (often called the 'radio frequency', or RF).
- Unfortunately, this cannot be done in general, as high frequency components are much more expensive, and generally perform more poorly than low frequency components.
- Thus, most radio interferometers use 'down-conversion' to translate the radio frequency information from the 'RF' to a lower frequency band, called the 'IF' in the jargon of our trade.
- For signals in the radio-frequency part of the spectrum, this can be done with almost no loss of information.
- But there is an important side-effect from this operation in interferometry which we now review.

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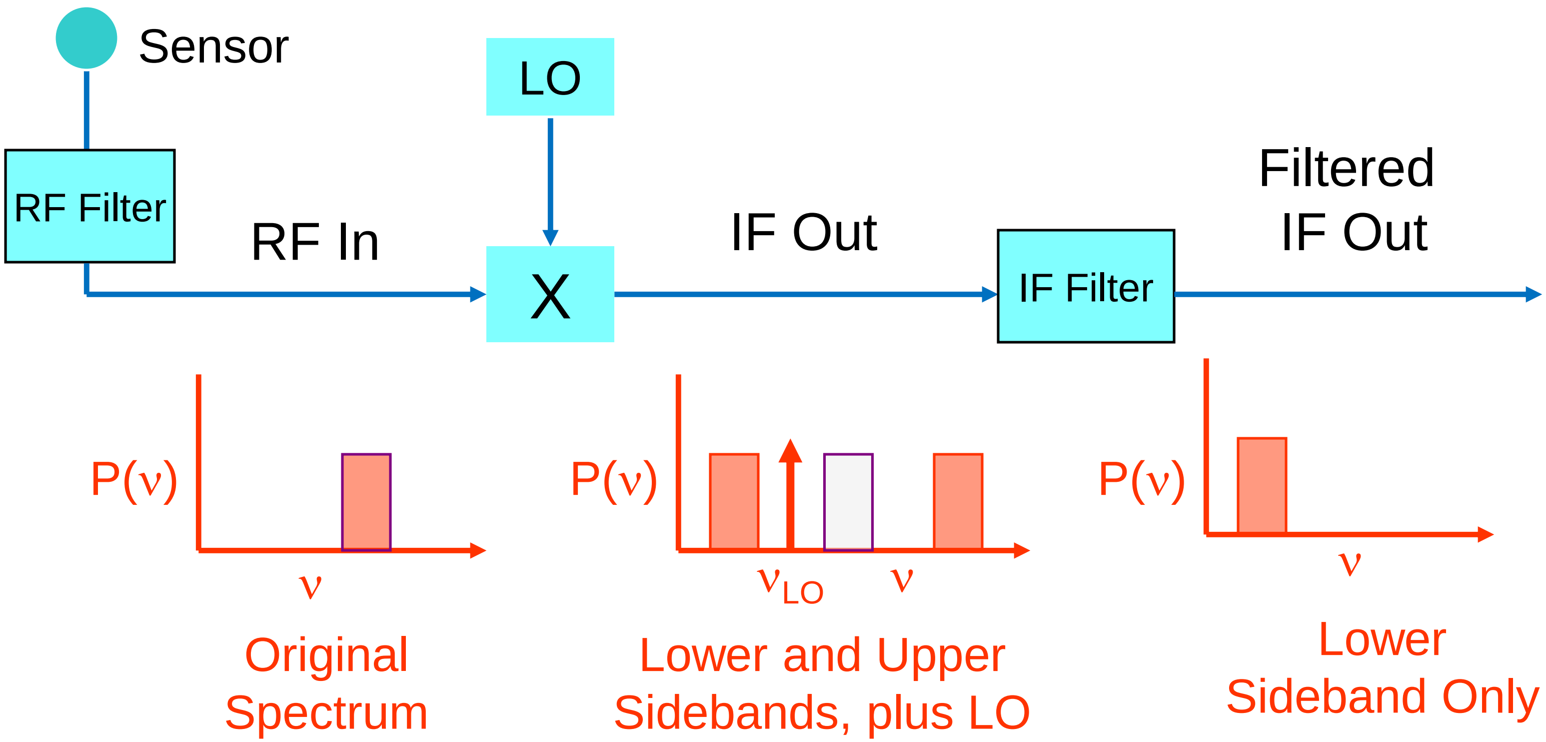
Downconversion

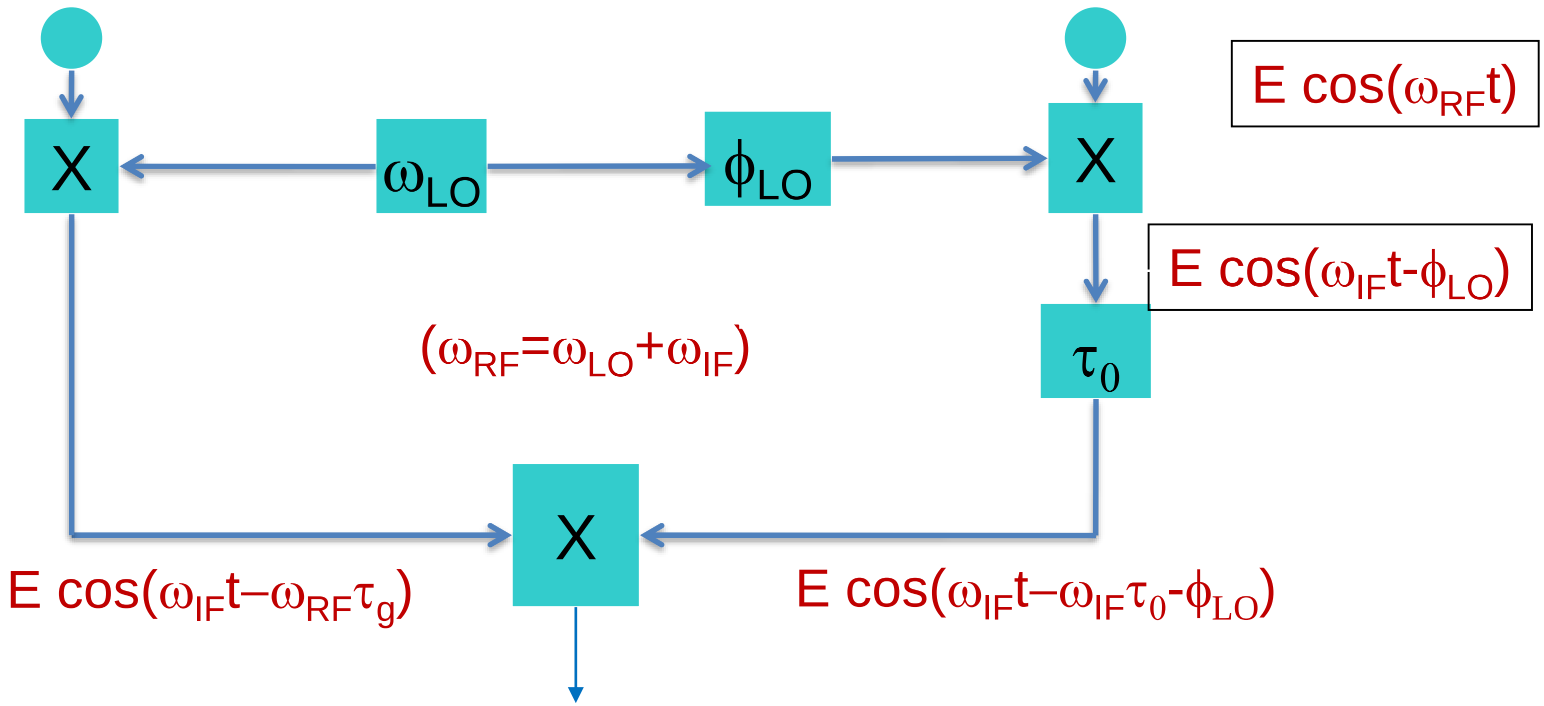
At radio frequencies, the spectral content within a passband can be shifted – with almost no loss in information, to a lower frequency through multiplication by a ‘LO’ signal.



This operation preserves the amplitude and phase relations

Signal Relations, with LO Downconversion

- The RF signals are multiplied by a pure sinusoid, at frequency ν_{LO}
- We can add arbitrary phase ϕ_{LO} on one side.



$$V = E^2 e^{-i(\omega_{RF}\tau_g - \omega_{IF}\tau_0 - \phi_{LO})}$$

Recovering the Correct Visibility Phase

- Unfortunately, this downconversion process changes the phase of the correlated product. To see this, note:
- The correct phase (RF interferometer) is:

$$\omega_{RF}\left(\tau_g - \tau_0\right)$$

- The observed phase, with frequency downconversion is:

$$\omega_{RF}\tau_g - \omega_{IF}\tau_0 - \phi_{LO}$$

- These will be the same when the LO phase is set to:

$$\phi_{LO} = \omega_{LO}\tau_0$$

- Thus, to retrieve the correct phase, we must adjust the LO phase.
- This is necessary because the delay, τ_0, has been added in the IF portion of the signal path, rather than at the RF frequency at which the delay actually occurs.
- The phase adjustment of the LO compensates for the delay having been inserted at the IF , rather than at the RF.

The Three 'Centers' in Interferometry

- You are forgiven if you're confused by all these 'centers'.
- So let's review:

1. **Beam Tracking (Pointing) Center**: Where the antennas are pointing to. Normally follows the source over time. (But doesn't have to …)
2. **Delay Tracking Center:** The location for which the delays are being set for maximum wide-band coherence.
3. **Phase Tracking Center:** The location for which the system phase has been adjusted in order to track the coherence (fringe) pattern position.

- Note: Generally, we make all three the same – but in general, the phase and delay centers are separable.

Geometry

- This is the second half. Here we cover:
- Coordinate systems
  - Direction cosines
  - (u,v) planes, and (u,v,w) volumes
  - 2-D ('planar') interferometers
  - 3-D ('volume') interferometers
  - Handling imaging with 'Volume' interferometers
- U-V Coverage, Visibilities, and Simple Structures.
- Illustrative examples of Visibilities.

Interferometer Geometry

- We have not defined any geometric system for our relations.
- The visibility functions we defined were generalized in terms of the scalar product between two fundamental vectors:
  - The baseline '**b**', defining the orientation and separation of the antennas, and
  - The unit vector '**s**', specifying the direction on the sky.
- The relationship between the interferometer's measurements, and the sky emission is:

$$\mathcal{V}_\nu(\mathbf{b}) = R_C - iR_S = \iint A_\nu(\mathbf{s}) I_\nu(\mathbf{s}) e^{-2\pi i\, \mathbf{b}\cdot\mathbf{s}/\lambda} d\Omega$$

- I now define a geometric coordinate frame for the interferometer.

The 2-Dimensional Interferometer

Case A: A 2-dimensional measurement plane.

- Suppose the measurements of $V_\nu(\mathbf{b})$ are taken entirely on a plane.
- Then a considerable simplification occurs if we arrange the coordinate system so one axis is normal to this plane.
- Let (u,v,w) be the coordinate axes, with w normal to this plane. Then the baseline's components are

$$\mathbf{b} = (\lambda \mathrm{u}, \lambda \mathrm{v}, \lambda \mathrm{w}) \rightarrow (\lambda \mathrm{u}, \lambda \mathrm{v}, 0)$$

u, v, and w are always measured in wavelengths.

- The components of the unit direction vector, **s**, are:

$$\mathbf{s} = (l, m, n) = \left(l, m, \sqrt{1 - l^2 - m^2}\right)$$

the simplification arises since |**s**|=1. Only two coordinates are needed to specify direction.

- (l,m,n) are the **direction cosines**.

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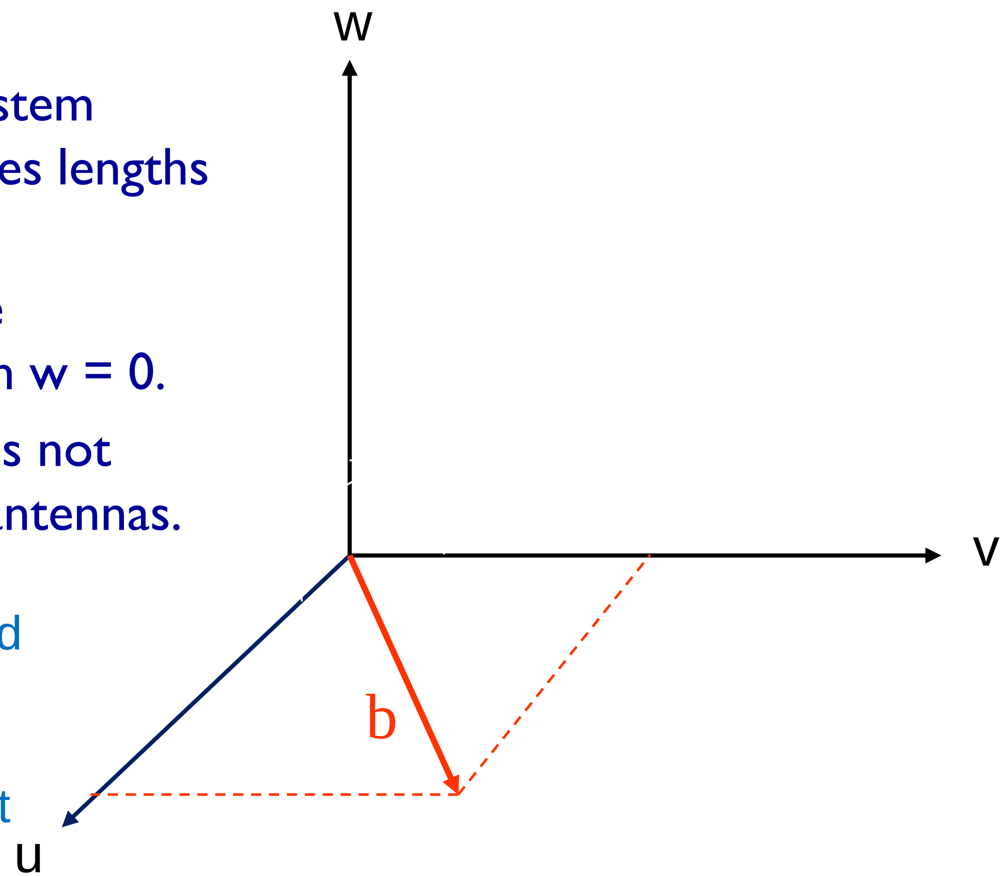
The (u,v,w) Coordinate System.

- Pick a cartesian coordinate system (u,v,w) to describe the baselines lengths and orientations.
- Orient this frame so the plane containing the baselines lies on w = 0.
- This is a spatial frame, but does not describe the locations of the antennas.



The baseline vector **b** is specified by its coordinates (u,v,w) (measured in wavelengths). In the case shown, w = 0, so that

$$\mathbf{b} = (\lambda u, \lambda v, 0)$$

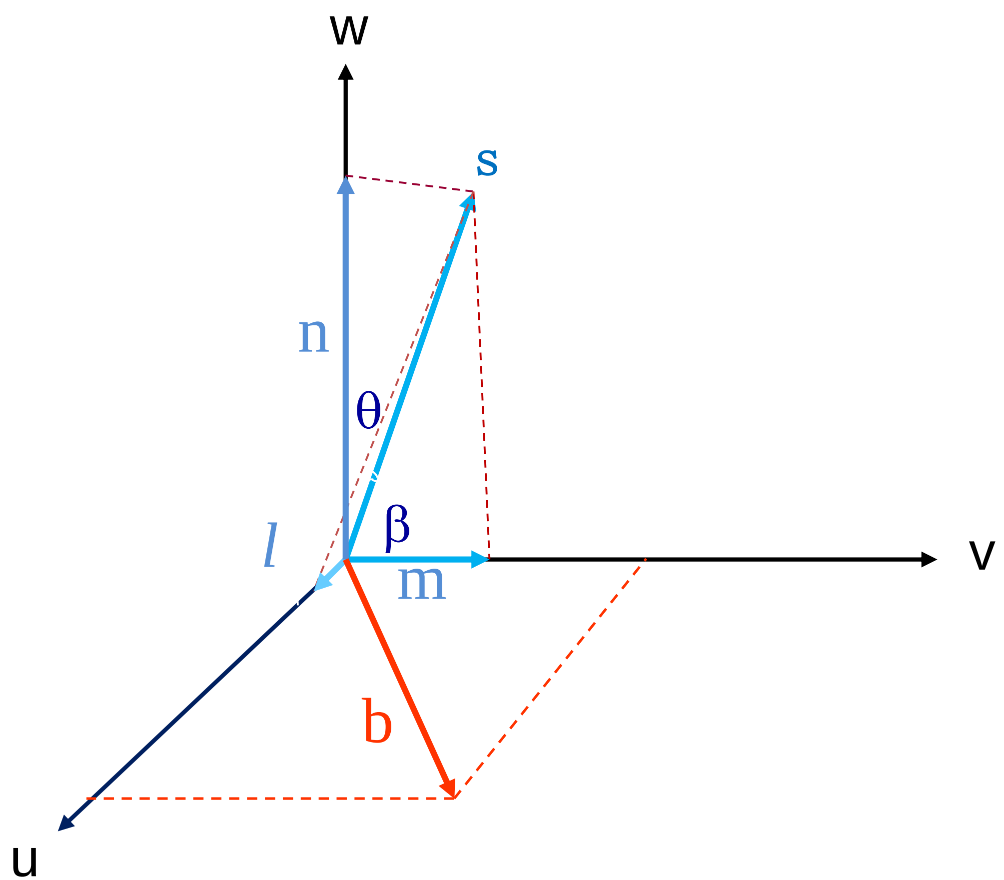
Direction Cosines – describing the source direction

The unit direction vector **s** is defined by its projections (l,m,n) on the (u,v,w) axes. These components are called the **Direction Cosines**.

$$l = \cos(\alpha)$$

$$m = \cos(\beta)$$

$$n = \cos(\theta) = \sqrt{1 - l^2 - m^2}$$



The angles, α, β, and θ are between the direction vector and the three axes.

The 2-d Fourier Transform Relation

Then, **b.s**/λ = u*l* + vm + wn = u*l* + vm, (because w = 0) from which we find,

$$V_\nu(u,\mathrm{v}) = \iint I(l,m) e^{-i2\pi(ul+\mathrm{v}m)}\, dl\, dm$$

which is a **2-dimensional Fourier transform** between the brightness and the spatial coherence function (visibility):

$$I_\upsilon(l,m) \Leftrightarrow V_\upsilon(u,\mathrm{v})$$

And we can now rely on two centuries of effort by mathematicians on how to invert this equation, and how much information we need to obtain an image of sufficient quality.

Formally,

$$I_\nu(l,m) = \iint V_\nu(u,\mathrm{v}) e^{i2\pi(ul+\mathrm{v}m)}\, du\, d\mathrm{v}$$

In physical optics, this is known as the 'Van Cittert-Zernicke Theorem'.

How we actually do this inversion is left to the 'Imaging' lecture.

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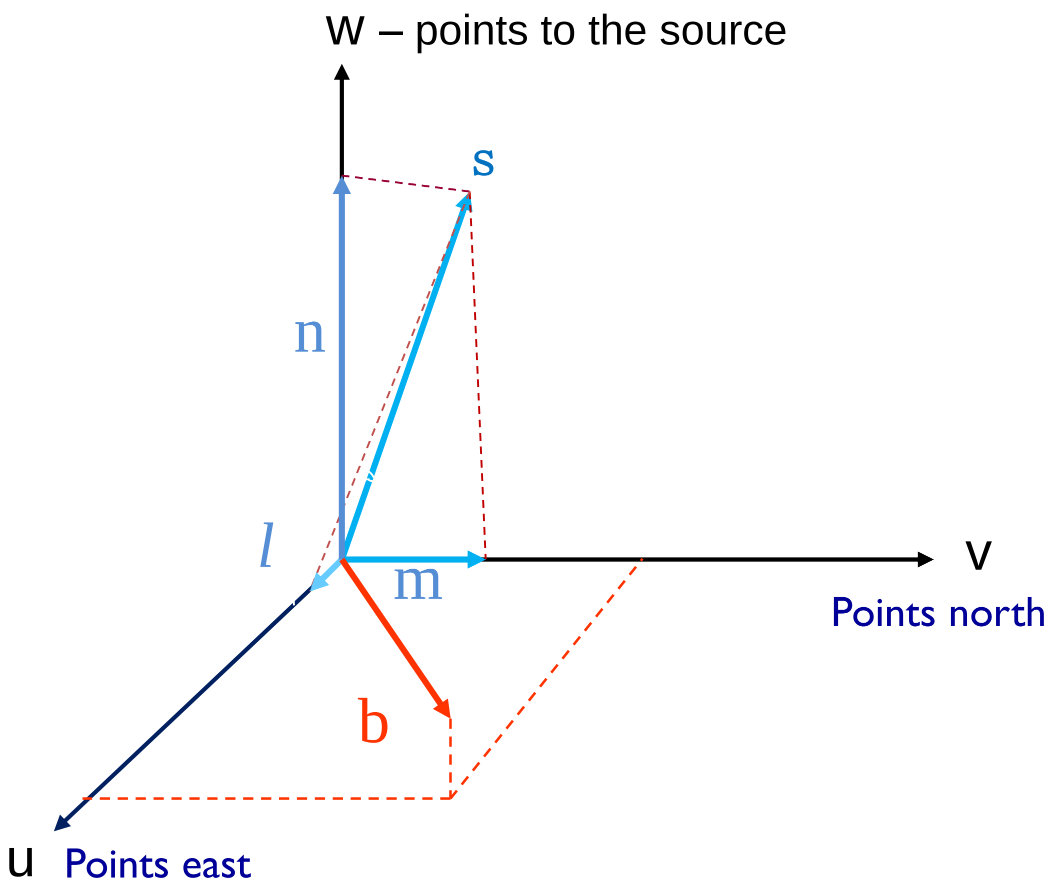
Interferometers with 2-d Geometry

- **Which interferometers can use this special geometry?**
  - a) Those whose baselines, over time, lie on a plane (any plane).
    - All E-W interferometers are in this group. For these, the w-coordinate points to the NCP. The (u,v) plane is parallel to the Equatorial Plane.
      - WSRT (Westerbork Synthesis Radio Telescope) (now shut down).
      - ATCA (Australia Telescope Compact Array) (E-W arm)
      - Cambridge 5km (Ryle) telescope (approximately).
  - b) Any coplanar 2-dimensional array, at a single instance of time.
    - In this case, the 'w' coordinate points to the zenith.
      - VLA, MeerKAT, and GMRT in snapshot (single short observation) mode.
- **What's the 'downside' of 2-d (u,v) coverage?**
  - Resolution degrades for observations that are not in the w-direction.
    - E-W interferometers have no N-S resolution for observations at the celestial equator.
    - A VLA snapshot of a source near the horizon will have no 'vertical' resolution.

Generalized Baseline Geometry

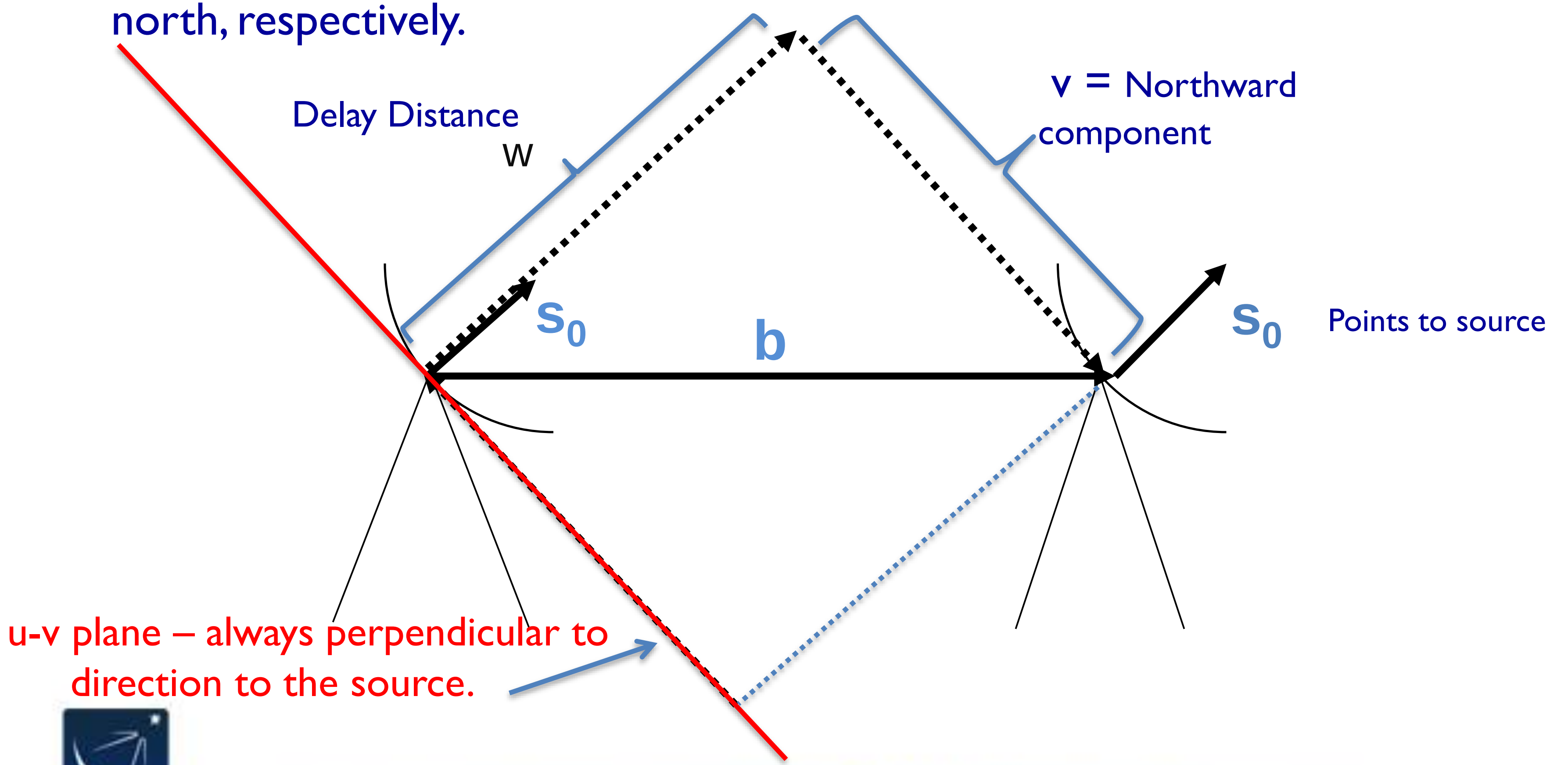
- General arrays (like the VLA) cannot use the 2-d geometry, since the antennas are not along an E-W line.
- Over time, their baselines move through a (u,v,w) volume
- In this case, we adopt a more general description, where all three baseline components are to be considered.
- Arrange 'w' to point to the source (phase tracking center), and orient the (u,v) plane so the 'v' axis points towards the north, and 'u' towards the east.



Baseline vector **b** now has three time-variable components.

General Coordinate System

- This is the coordinate system in most general use for synthesis imaging.
- **w** points to, and follows the source, **u** towards the east, and **v** towards the north. The direction cosines *l* and m then increase to the east and north, respectively.



3-d Interferometers

Case B: A 3-dimensional measurement volume:

- The complete relation between the visibility and sky brightness, for a phase-tracking interferometer, is now more complicated:

$$V_\nu(u,v,w) = \iint I_\nu(l,m) e^{-2i\pi[ul+vm+w(\sqrt{1-l^2-m^2}-1)]} dl dm$$

(Note that this is neither a 2-D or a 3-D Fourier Transform).

- The problem lies with the third term: $w\left(1-\sqrt{1-l^2-m^2}\right) \approx w\theta^2/2$
- Where $\theta = \sqrt{l^2+m^2}$ is the offset to the map edge, in radians, and w is the `depth' of the (u,v,w) sampled volume, in wavelengths.
- If this term is very small (<<1), then we might ignore it, in which case we return to a nice 2-D transform.

$$V_\nu(u,v) = \iint I_\nu(l,m) e^{-2i\pi[ul+vm]} dl dm$$

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The Problem with Non-coplanar Baselines

- Use of the 2-D transform for non-coplanar interferometer arrays (like the VLA, when used over time) always results in an error in the images.
- We can derive an easy criterion for use of the 2-D transform by noting the maximum angle for imaging is limited by the primary beam to:

$$\theta_{\max} \sim \lambda / D$$

- While the maximum 'depth' of the (u,v,w) volume is

$$w_{\max} = B / \lambda$$

- This gives us the 'Clark' condition for using

a 2-d transform

$$\frac{\lambda B}{D^2} < 1$$

  - The problem is most acute for small-diameter antennas (D small) long baselines (B large), and long wavelengths (λ large) – low frequency arrays with small antennas.
  - Note: Low elevation observations (large w) should be avoided!

PSF	FOV	Ratio
0.1”	3.4’	2030
0.3”	5.9’	1172
1.0”	10.7’	642
3.0”	18.6’	371
10”	34’	203
30”	59’	117
100”	108’	64

Table showing the maximum FOV for a given resolution.

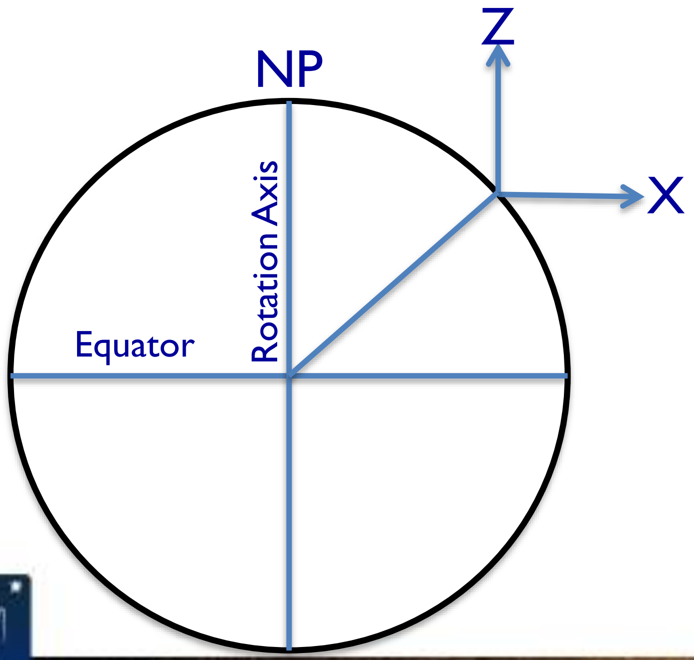
Four Solutions for Non-Coplanar Arrays

1. Do a 3-D transform. This is the **only** correct solution.
   - Grid data in 3-dimensions, and transform to a 3-d cube.
   - The sky appears on a sphere of unit radius. (Cool!)
   - Not a practical solution (~90% of computed cells are empty).
2. If the array is instantaneously coplanar, sum up a series of snapshots.
   - Requires re-projection of each image's coordinates
   - Deconvolution problematic, due to limited sensitivity
3. Do facetted imaging – partition the field of view into lots of little images, each of which meets the small-angle criterion.
   - Requires phase offsets and recomputation of baselines for each facet
   - Can apply 'local' calibration for each facet.
4. Project the visibilities onto the w = 0 plane. ('W-Projection').
   - Effectively makes the array fully coplanar.

- These important issues are discussed in detail in Preshanth's talk.

Coverage of the U-V Plane

- Obtaining a good image of a source requires adequate sampling ('coverage') of the (u,v) plane.
- Adopt an earth-based coordinate grid to describe the antenna positions:
  - X points to H=0, δ=0 (intersection of meridian and celestial equator)
  - Y points to H = -6, δ = 0 (to east, on celestial equator)
  - Z points to δ = 90 (to NCP).



- Thus, Bx, By are the baseline components in the Equatorial plane,
- Bz is the baseline component along the earth's rotation axis.
- All components in wavelengths and do not change in time.
- Now compute the (u,v,w) components of the baseline for a given H (hour angle) and δ.

NRAO

(u,v,w) Coordinates

- Then, it can be shown that

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} \sin H_0 & \cos H_0 & 0 \\ -\sin\delta_0 \cos H_0 & \sin\delta_0 \sin H_0 & \cos\delta_0 \\ \cos\delta_0 \cos H_0 & -\cos\delta_0 \sin H_0 & \sin\delta_0 \end{pmatrix} \begin{pmatrix} B_X \\ B_Y \\ B_Z \end{pmatrix}$$

- The u and v coordinates describe E-W and N-S components of the **projected** interferometer baseline.
- The w coordinate is the delay distance in wavelengths between the two antennas. The geometric time delay, τ_g is given by

$$\tau_g = \frac{\lambda}{c} w = \frac{w}{\upsilon}$$

- The time derivative of w, called the fringe frequency ν_F is

$$\nu_F = \frac{dw}{dt} = -\frac{dH}{dt} u \cos\delta_0 = -\omega_E u \cos\delta_0$$

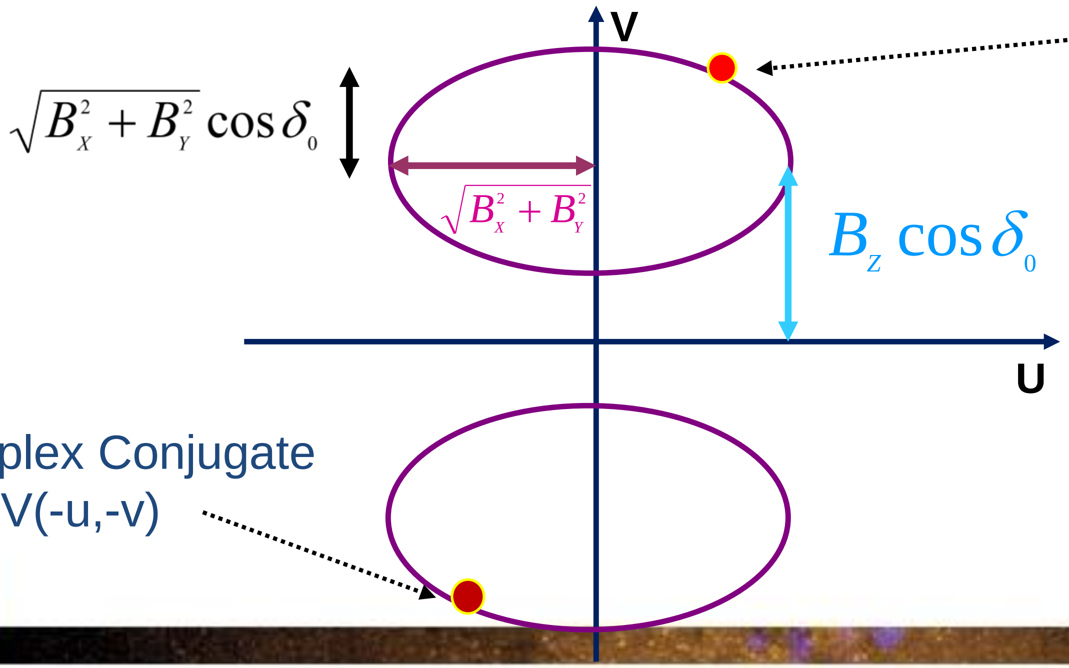
Baseline Locus – the General Case

- Each baseline, over 24 hours, traces out an ellipse in the (u,v) plane:

$$u^2 + \left(\frac{v - B_Z \cos\delta_0}{\sin\delta_0} \right)^2 = B_x^2 + B_y^2$$

- Because brightness is real, each observation provides us a second point, where: V(-u,-v) = V*(u,v)
- E-W baselines (Bx = Bz = 0) have no 'v' offset in the ellipses.

A single Visibility: V(u,v)



Good UV Coverage requires many simultaneous baselines amongst many antennas, or many sequential baselines from a few antennas.

Its Complex Conjugate V(-u,-v)

E-W Array Coverage and Synthesized Beams

- The simplest case is for E-W arrays, which give coplanar coverage.
- Then, Bx = Bz = 0, and B_y = B, the baseline length.
- For this, the (u,v,w) coordinates become especially simple:

$\mathrm{u} = B\cos H_0$	E-W component
$\mathrm{v} = B\sin\delta_0 \sin H_0$	N-S component
$\mathrm{w} = -B\cos\delta_0 \sin H_0$	Delay component

- The locus in the (u,v) plane is an ellipse centered at the origin.
- At δ = 90, w = 0, and the locus is a circle of radius B.
- At δ = 0, v = 0, and the locus is a line of length = B.

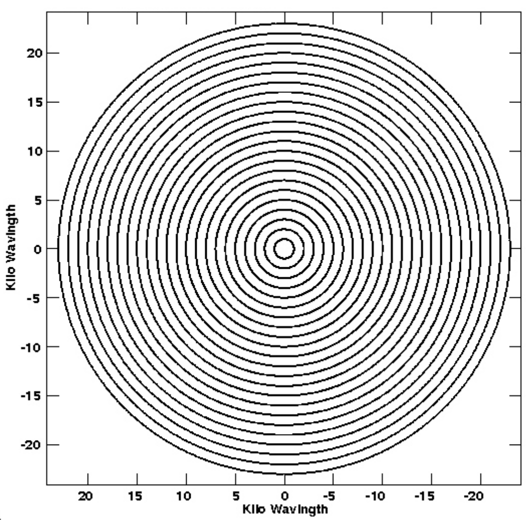
NRAO

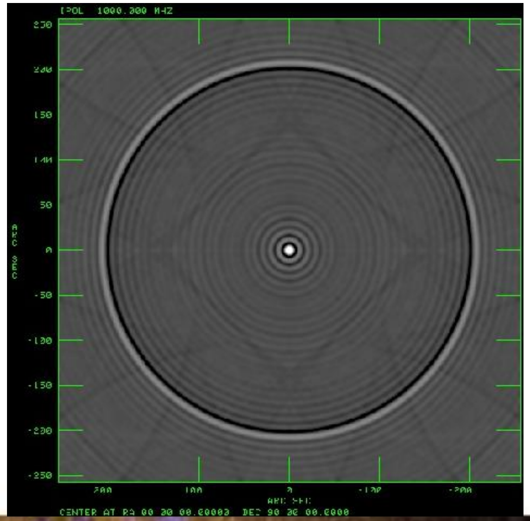
E-W Array (u,v) Coverage and Beams

- The simplest case is for E-W arrays, which give coplanar coverage.
- Then, Bx = Bz = 0, and the ellipses are centered on the origin.
- Consider a 'minimum redundancy array', with eight antennas located at

  0, 1, 2, 11, 15, 18, 21 and 23 km along an E-W arm.

- Of the 28 simultaneous spacings, 23 are of a unique separation.
- The U-V coverage (over 12 hours, including complex conjugate) at δ = 90, and the synthesized beam are shown below, for a wavelength of 1m.



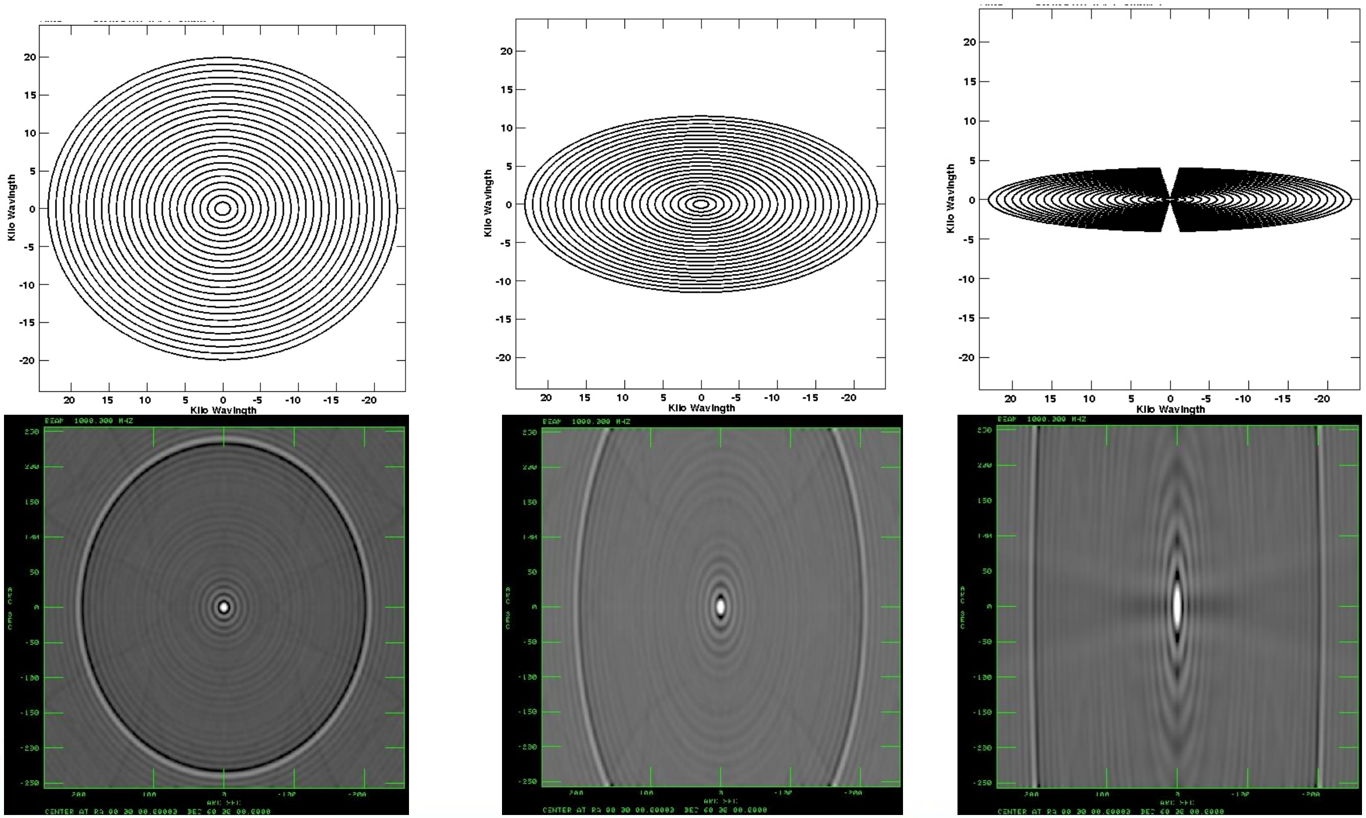


δ = 90

E-W Arrays and Low-Dec sources.

- But the trouble with E-W arrays is that they are not suited for low-declination observing.
- At δ=0, coverage degenerates to a line.

δ=60 δ=30 δ=10



U-V Coverage

PSF

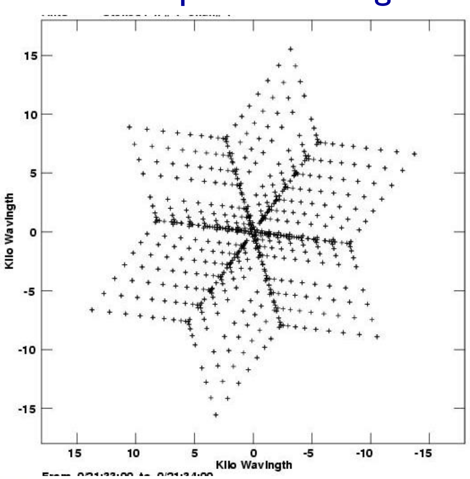
Getting Good Coverage near $\delta = 0$

- The only means of getting good 2-d angular resolution at all declinations is to build an array with N-S spacings.
- Many more antennas are needed to provide good coverage for such geometries.
- The VLA was designed to do this, using 9 antennas on each of three equiangular arms.
- Built in the 1970s, commissioned in 1980, the VLA vastly improved radio synthesis imaging at all declinations.
- Each of the 351 (=27*26/2) spacings traces an elliptical locus on the (u,v) plane.
- Every baseline has some (N-S) component, so none of the ellipses is centered on the origin.

Advantages of 2-d arrays

- The most obvious advantage is that the (u,v) coverage is instantaneously 2-dimensional.
- This means that a 2-d image of the emission can – in principle – be formed from short observations.
- By contrast, the E-W(`1-d') interferometer must observe over a 12-hour period in order to populate the (u,v) plane.
- A snapshot with an E-W interferometer gives a one-dimensional beam. (Not very useful).

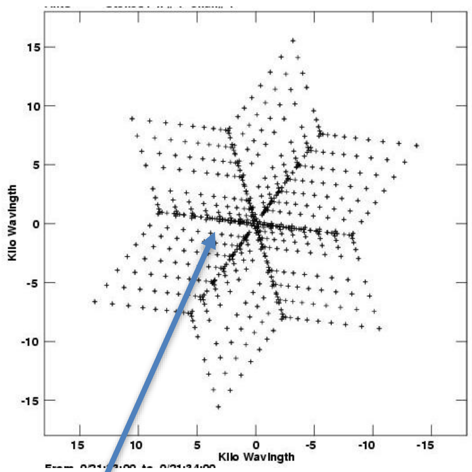
2-d snapshot coverage



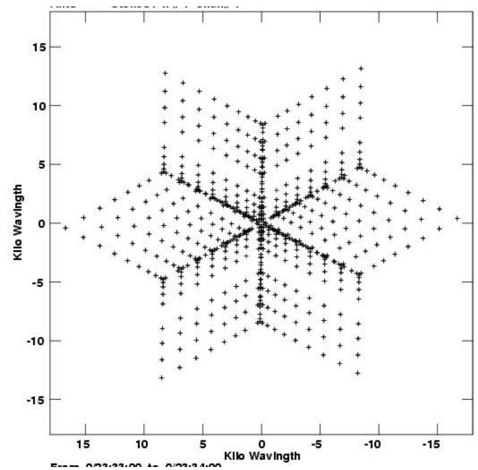
1-d snapshot coverage

Sample VLA (U,V) plots for 3C147 (δ = 50)

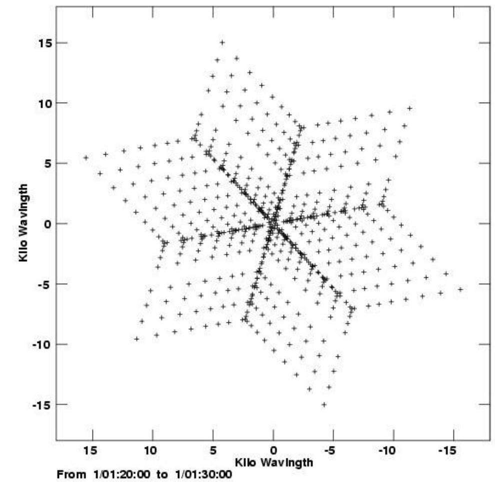
- Snapshot (u,v) coverage for HA = -2, 0, +2 (with 26 antennas).



HA = -2h

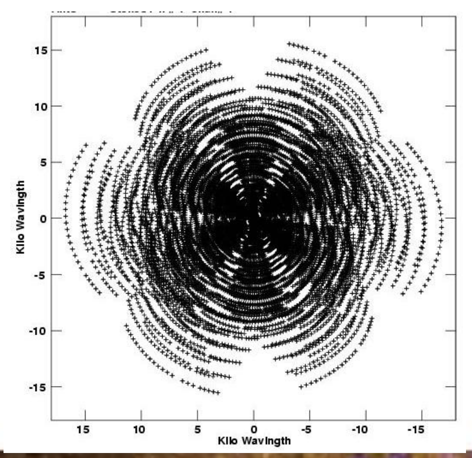


HA = 0h



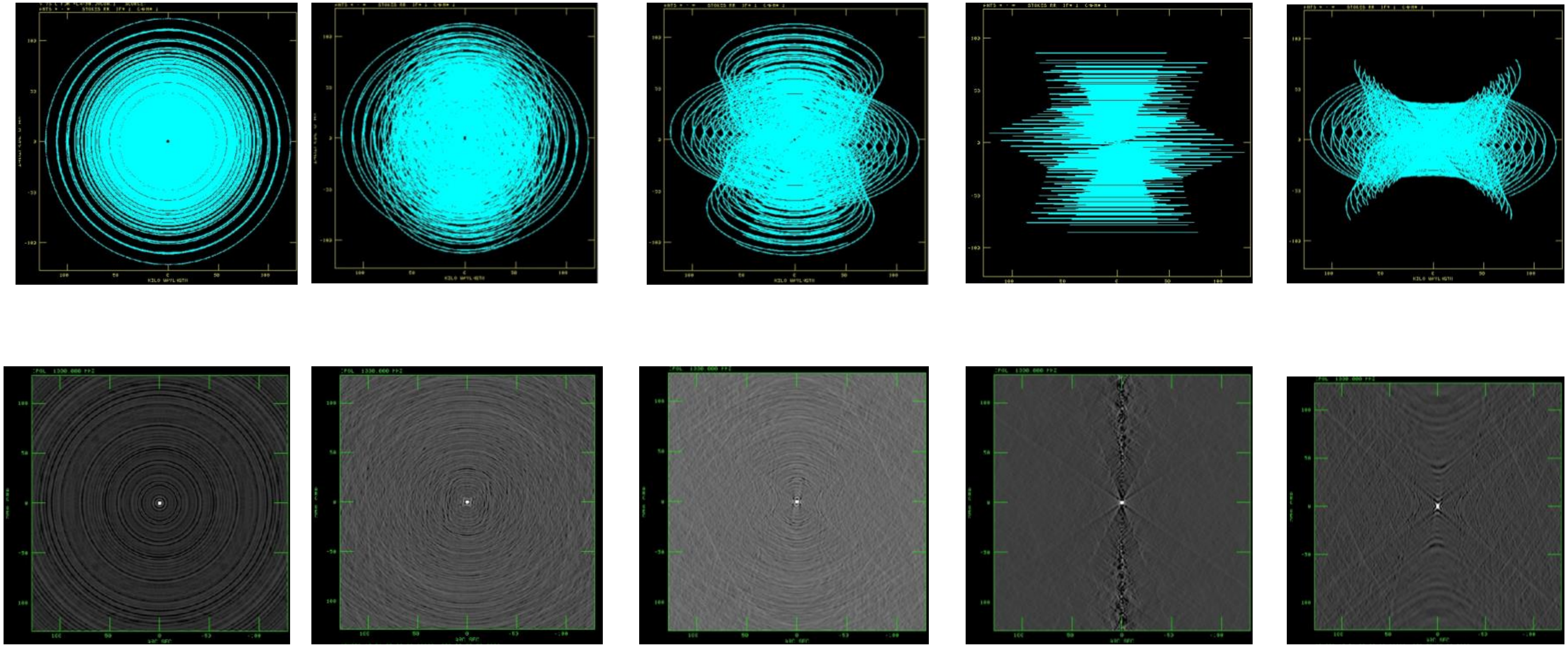
HA = 2h

Note the oversampling along six 'rays'. This causes trouble in the imaging …



Coverage over all four hours.

VLA Coverage and Beams



$\delta=90$ $\delta=60$ $\delta=30$ $\delta=0$ $\delta=-30$

- Good coverage at all declinations, but troubles near $\delta=0$ remain.

UV Coverage and Imaging Fidelity

- Although the VLA represented a huge advance over what came before, its UV coverage (and imaging fidelity) is far from optimal.
- The high density of samplings along the arms (the 6-armed star in snapshot coverage) results in 'rays' in the images due to small errors.
- A better design is to 'randomize' the location of antennas within the span of the array, to better distribute the errors.
- Of course, more antennas would really help! :) .
- The VLA's wye design was dictated by its 220 ton antennas, and the need to move them. Railway tracks were the only answer.
- New, and planned major arrays utilize smaller, lighter elements which must not be positioned with any linear regularity.
- Examples: ALMA, MeerKAT, ASCAP, ngVLA, SKA, DSA2000
- All of these have (or will have) superior imaging capability.

Examples with Real Data!

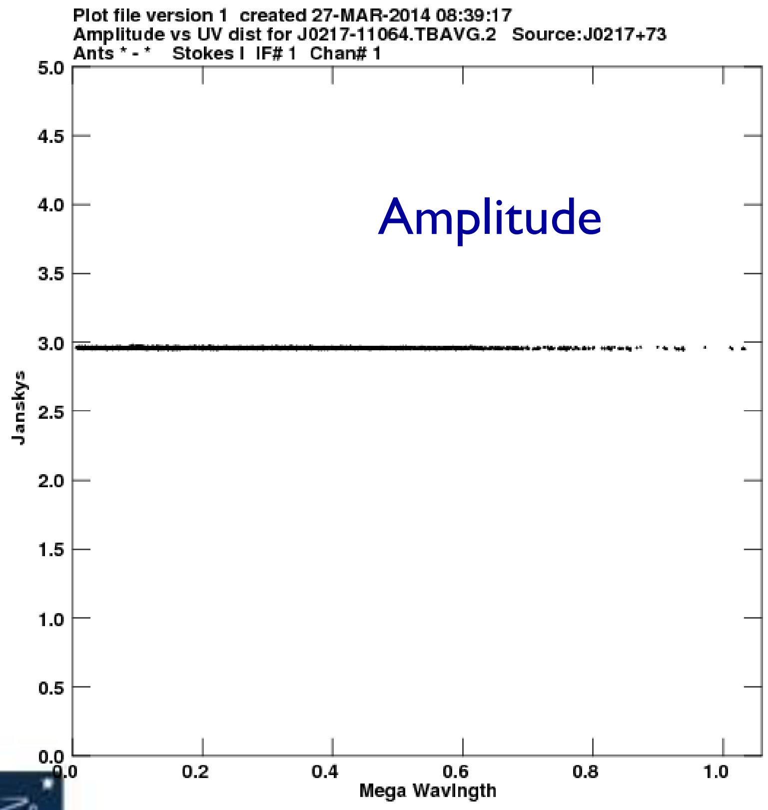
- Enough of the analysis!
- I close with some examples from real observations, using the VLA.
- These are two-dimensional observations (function of 'u' (EW) and 'v' (NS) baselines).
- Plotted are the visibility amplitudes version baseline length:

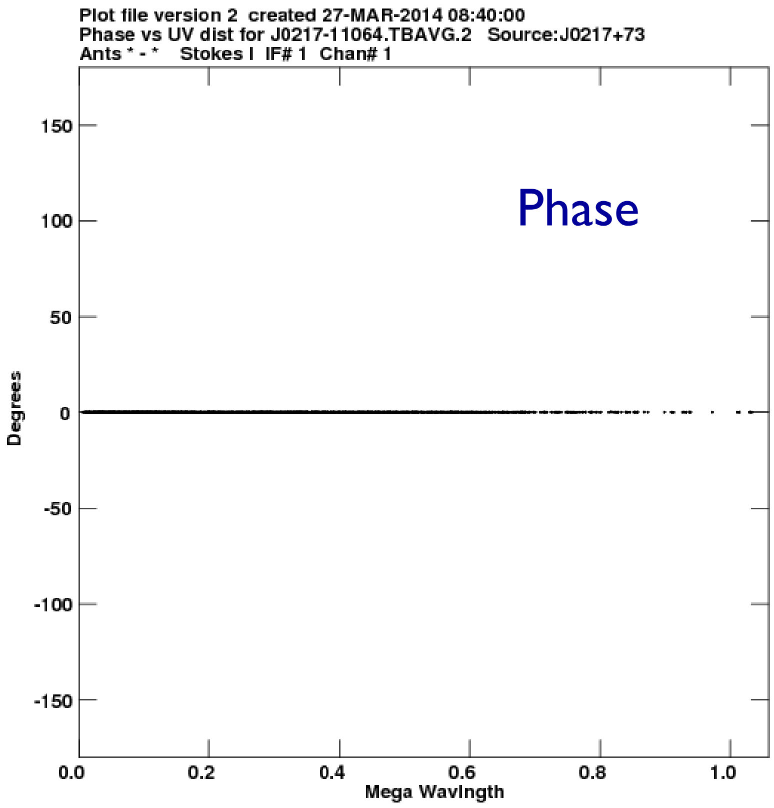
$$q = \sqrt{u^2 + v^2}$$

- Plotting visibilities in this way is easy, and often gives much information into source structure – as well as a diagnosis of various errors.

Examples of Visibilities – A Point Source

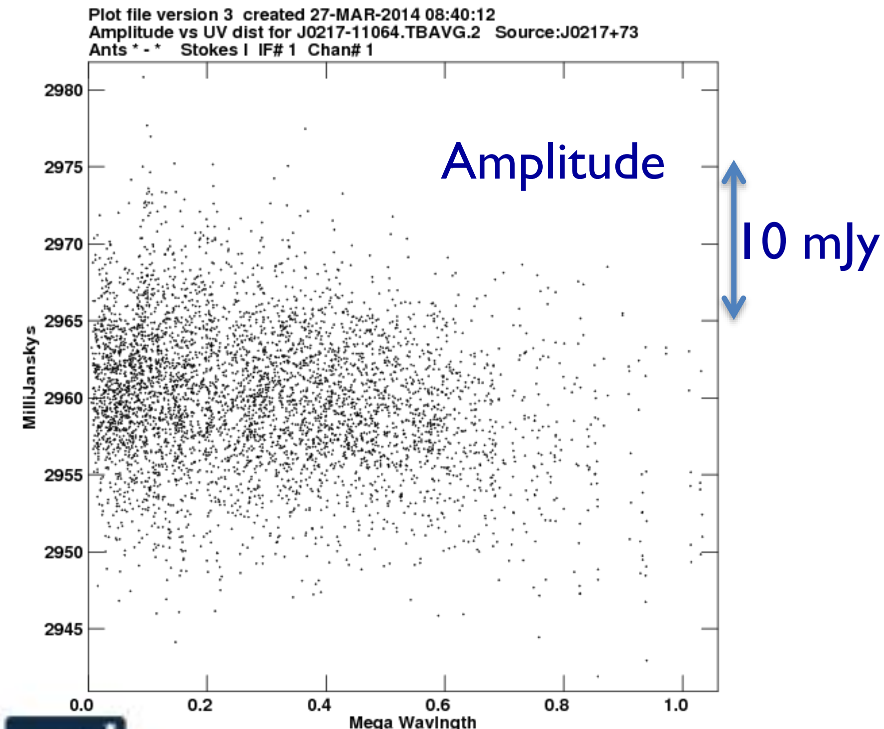
- Suppose we observe an unresolved object, at the phase center.
- What is its visibility function?

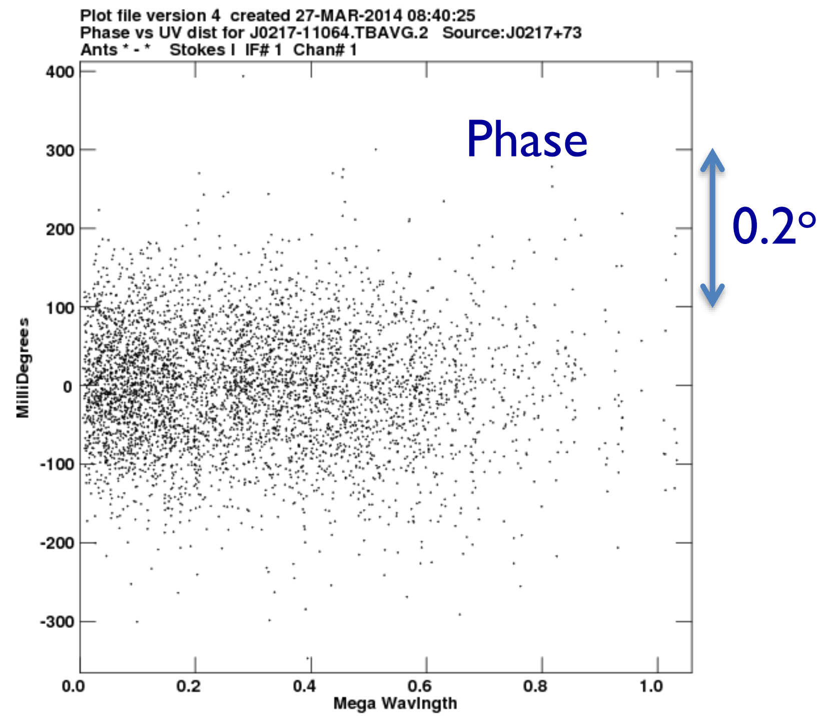




Zoom in to see the noise …

- The previous plots showed consistent values for all baselines.
- Zooming in shows the noise (and, possibly, additional structure).



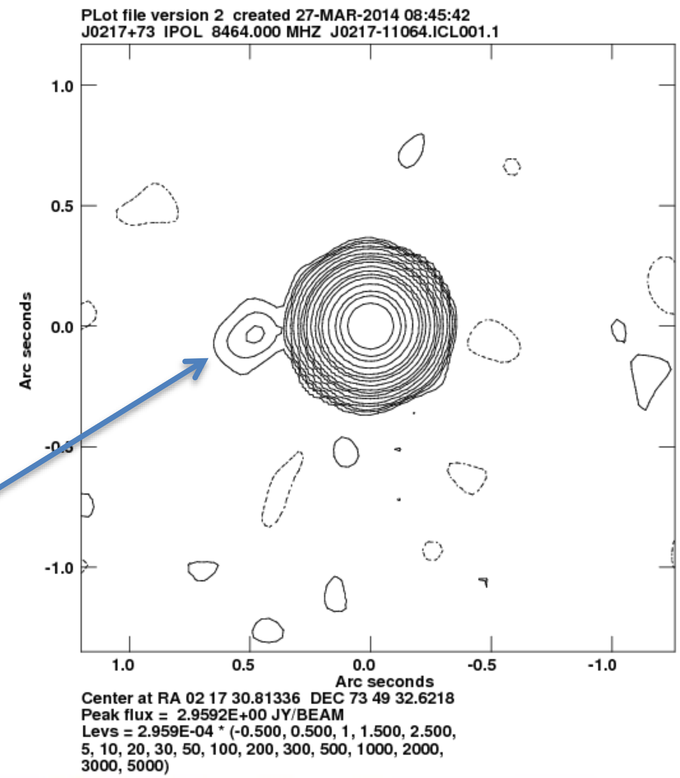


And the Map …

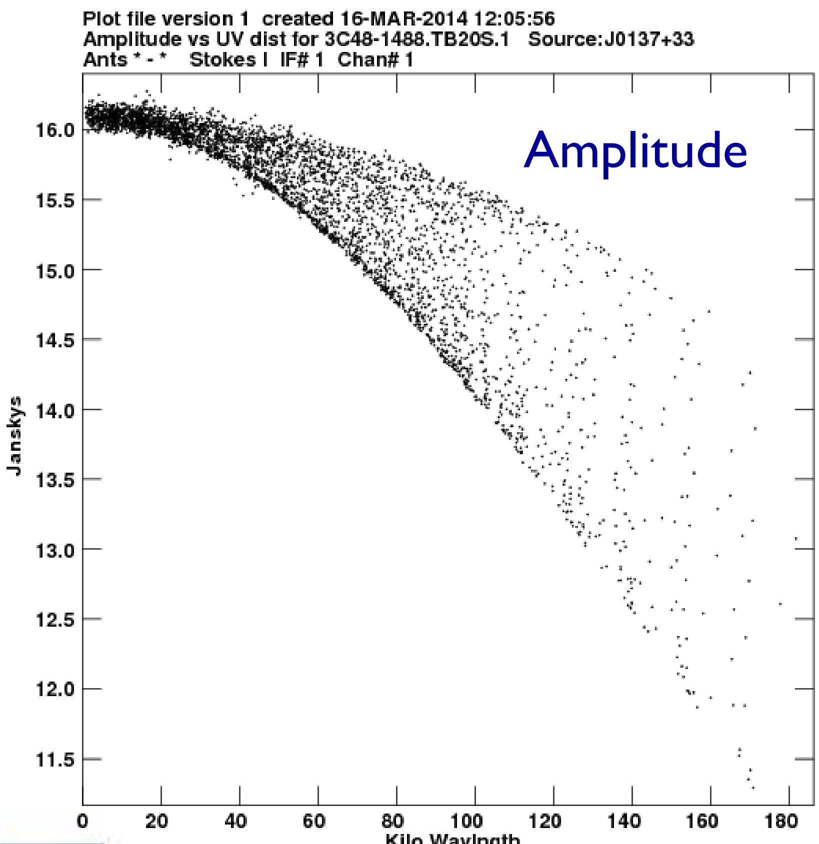
- The source is unresolved … but with a tiny background object.
- Dynamic range: 50,000:1.

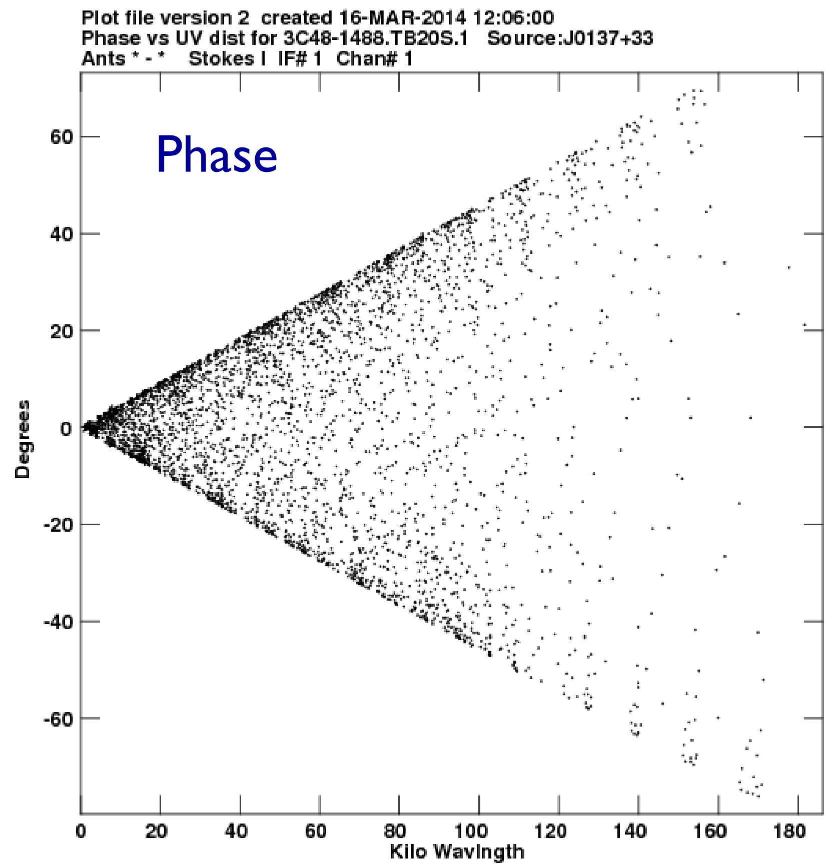
The flux in the weak nearby object is only 0.25 mJy – too low to be seen on any individual visibility.

Real!



3C48 at 21 cm wavelength – a slightly resolved object.



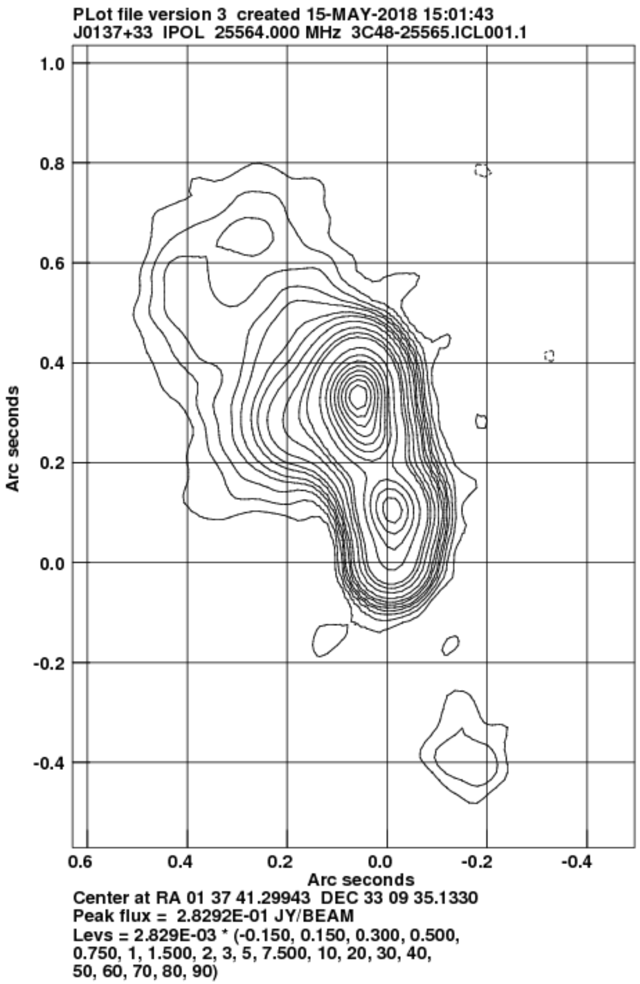


3C48 position and offset

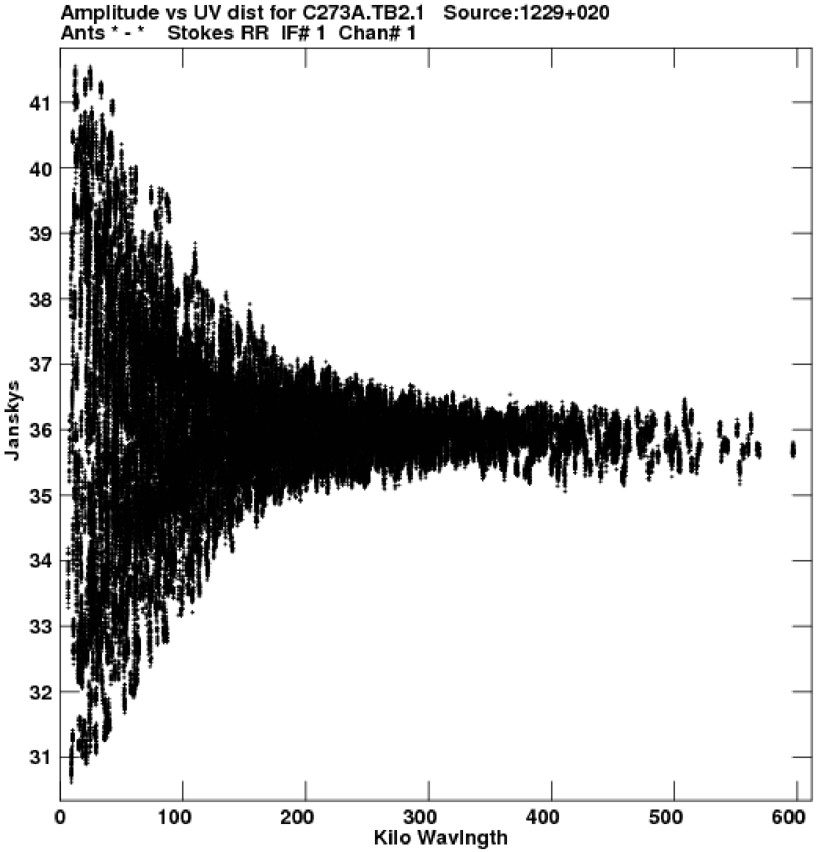
- You can learn quite a bit just by looking at the gross properties of the visibility amplitudes/phases, noting:
- **A 206265 wavelength baseline makes a 1 arcsecond fringe.**
- The linear phase slope is ~90 degrees over 180,000 wavelengths.
  - 90 degrees is ¼ of a fringe, and one fringe is one arcsecond. Thus, the source centroid is ~ 250 mas from the phase center.
- The amplitudes show slight (25%) resolution at 180,000 wavelengths. There is an upper and lower envelope.
  - The source is extended by a fraction (few tenths) of an arcsecond.
  - One axis has about a twice the size of the other.

3C48 Structure (at 25 GHz …)

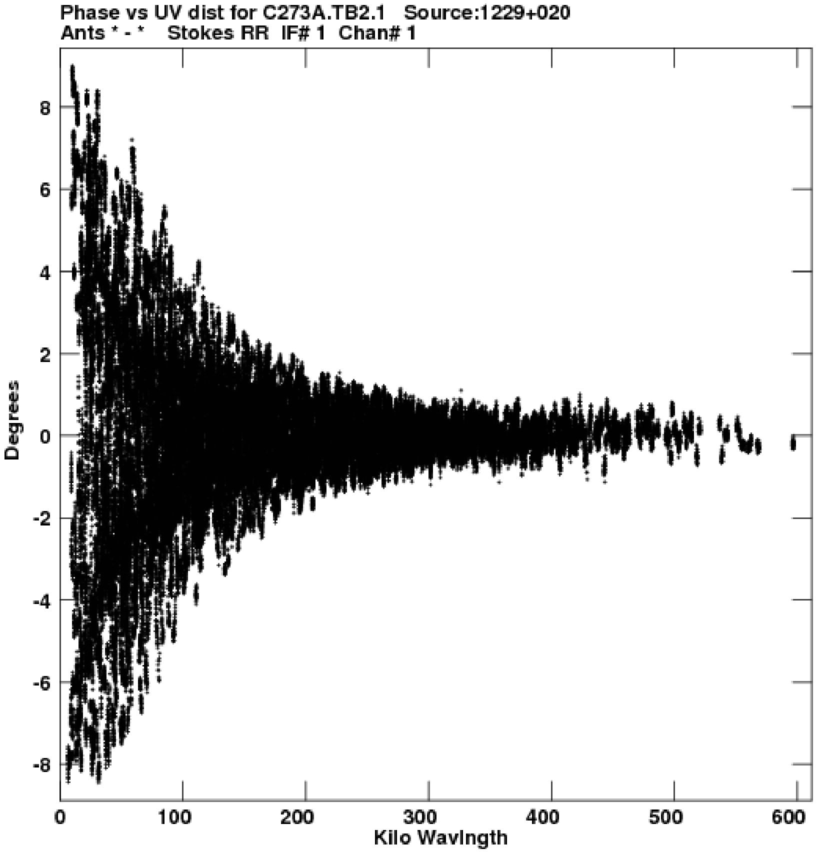
- The 1400 MHz image made from the data shown in the last slide doesn't show the structure well (poorly resolved).
- So here is the source at a higher frequency, where the resolution is 18 X higher (85 milliarcseconds)
- The emission centroid is about 250 milliarcseconds from the phase center, and less than 1 arccsecond in size, with roughly a 2:1 ratio in size.



3C273 – The first known quasar



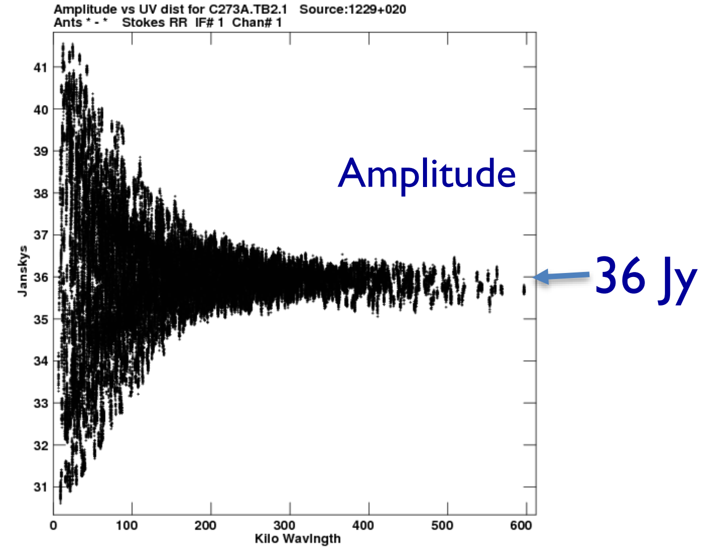
Visibility Amplitude

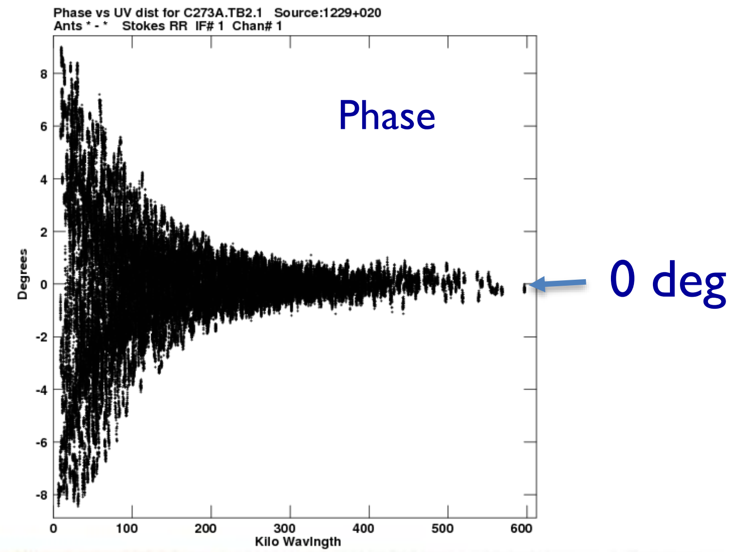


Visibility Phase

3C273 – jet size revealed …

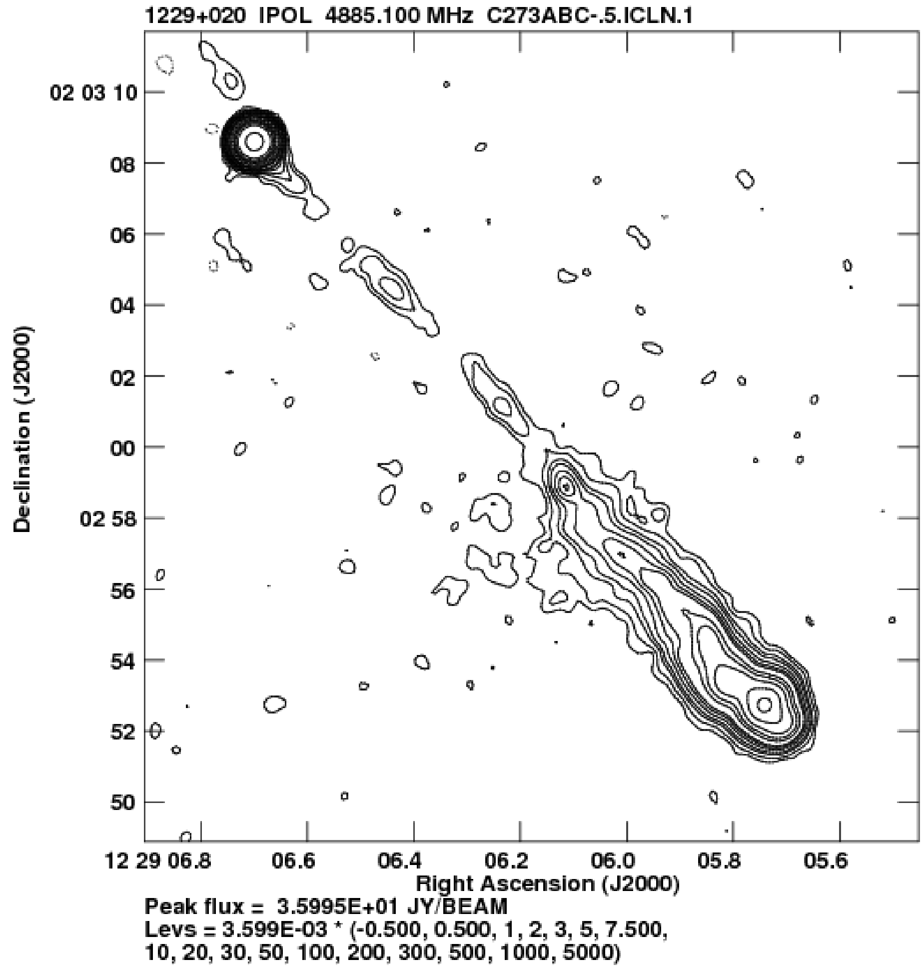
- The narrowing of the visibility amplitudes, with no sign of resolution on the long spacings, tells us:
  - There a 36 Jy unresolved `point'-source.
- The zero phase, and absence of a phase gradient on the long spacings tells us:
  - The point source is at the center
- The extended emission `resolves out' at ~100 -- 200 Kλ
- This indicates an extended structure with width of ~1 to 2"
- Rapid oscillations in both amplitude and phase tells us:
  - A much larger elongation is also present



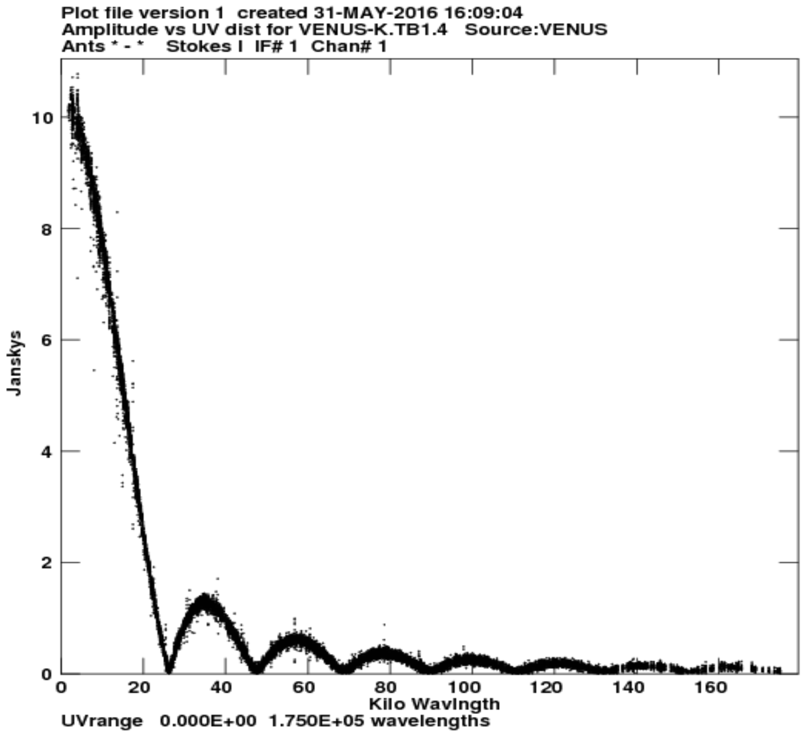


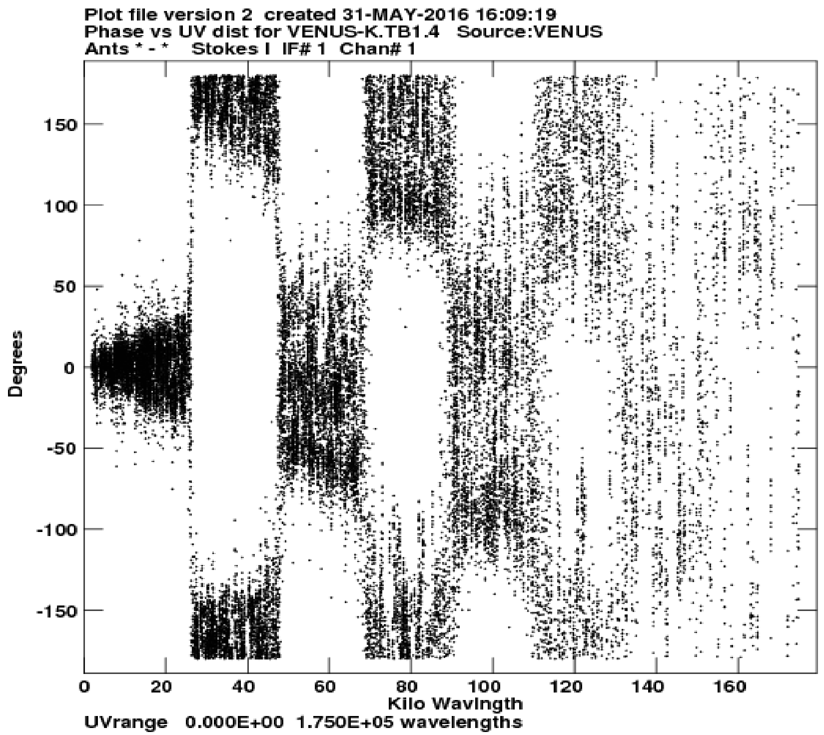
3C273 -- Image

- Actual structure revealed by making the image.
- There is a 36.0 Jy unresolved nucleus, with a one-sided jet.
- Jet width ~ 2 arcseconds
- Jet length ~ 18 arcsecond.



The Planet Venus at 19 GHz.

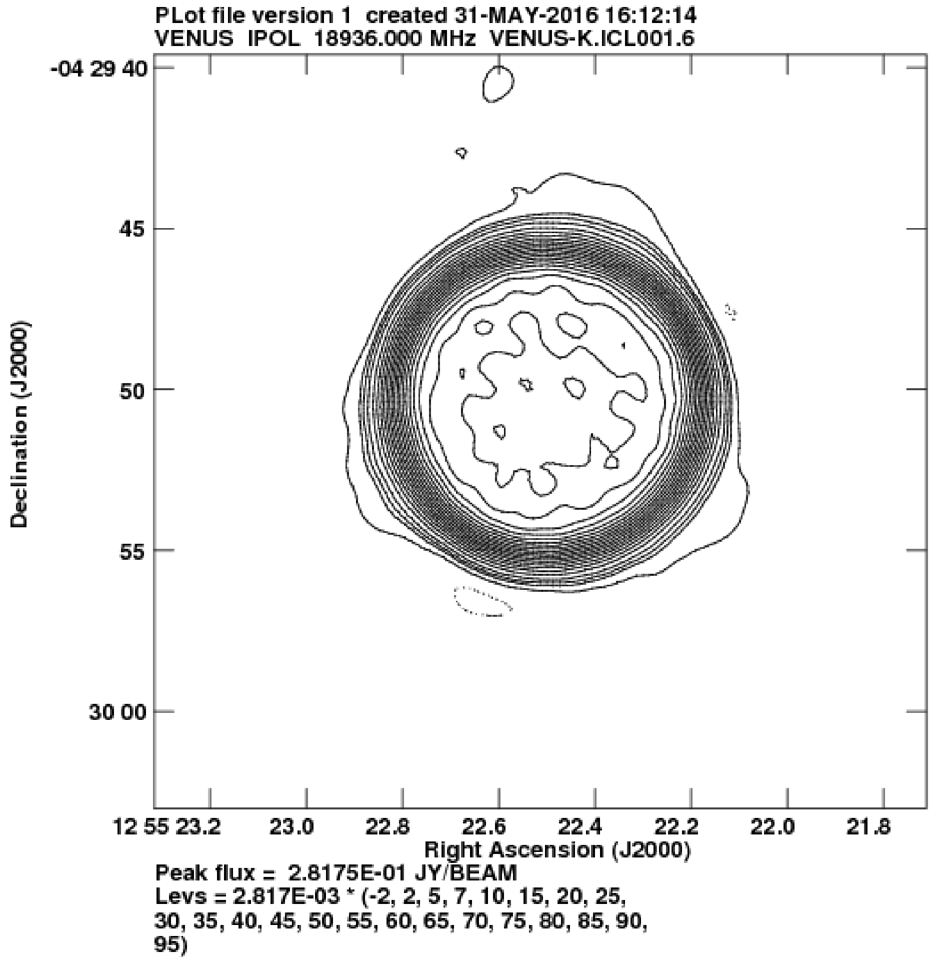




- The visibilities are circularly symmetric. The phases alternate between zero and 180 degrees.
- The source must be circularly symmetric and centered.
- The visibility null at 25 kλ indicates angular size of ~ 10 arcseconds.

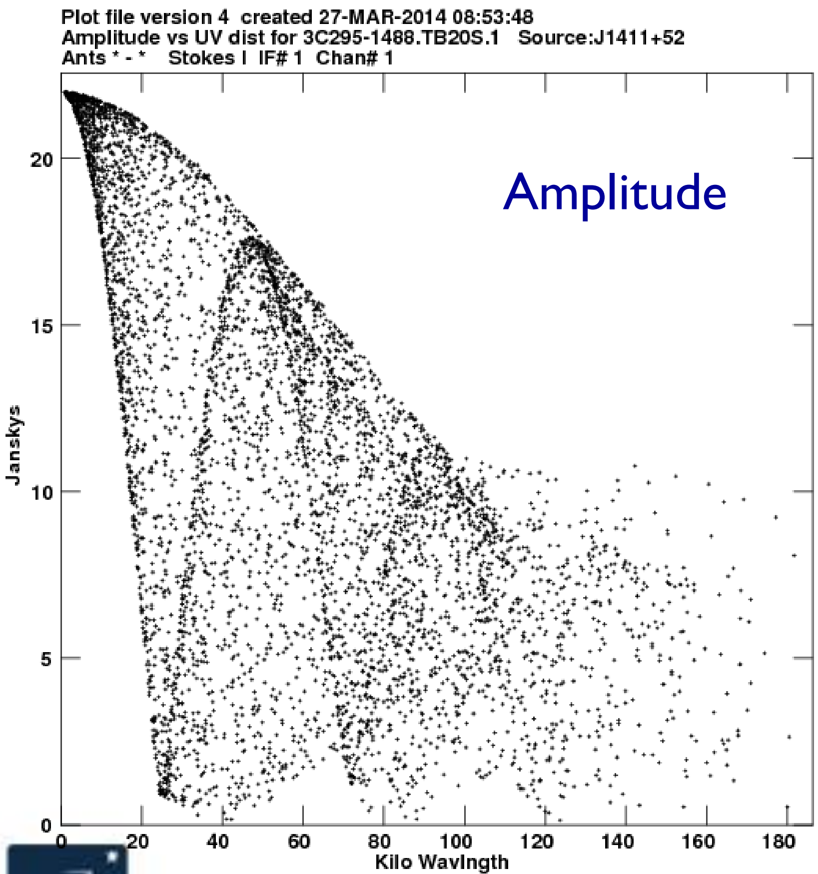
And the image looks like:

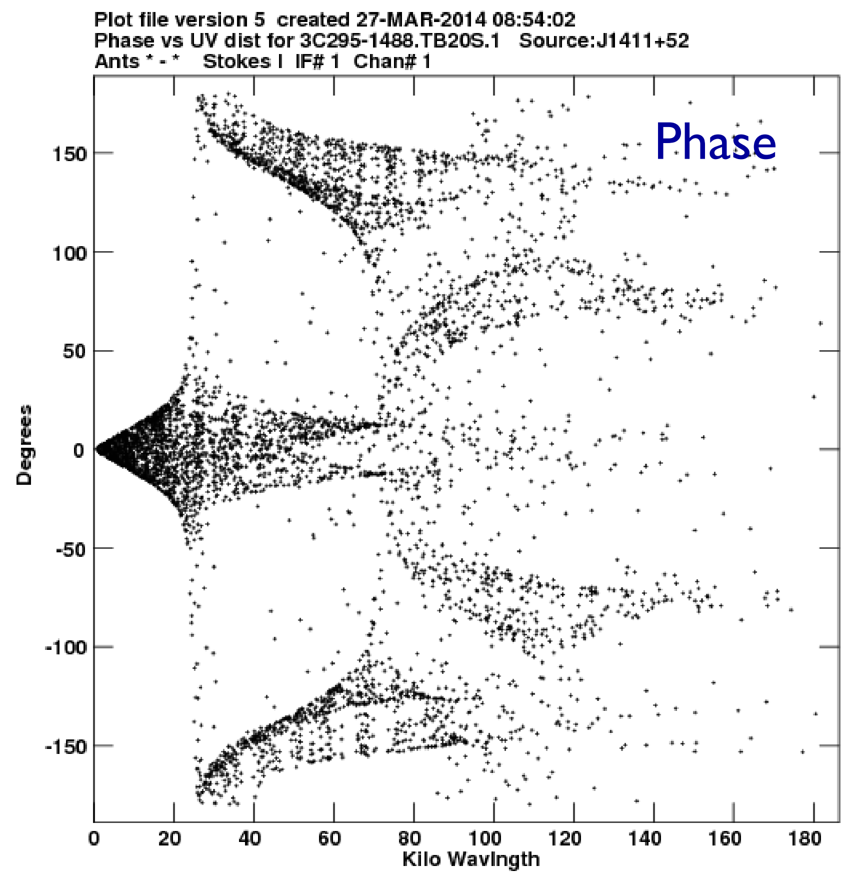
- It's a perfectly uniform, blank disk of diameter 10 arcseconds.
- The Visibility function, in fact, is an almost perfect Bessel function of zero order: $J_0(q)$.
- A perfect J_0 would arise from a perfectly sharp disk. Atmospheric opacity effects 'soften' the edge, resulting in small deviations from the J_0 function at large baselines.



3C295 – a Well Resolved Object

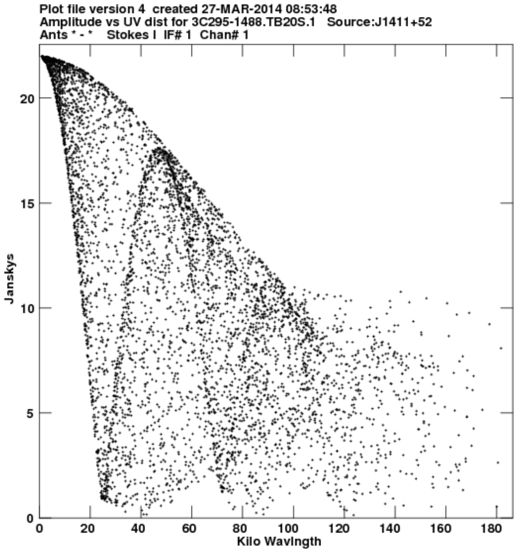
- The flux calibrator 3C295 at 1400 MHz.

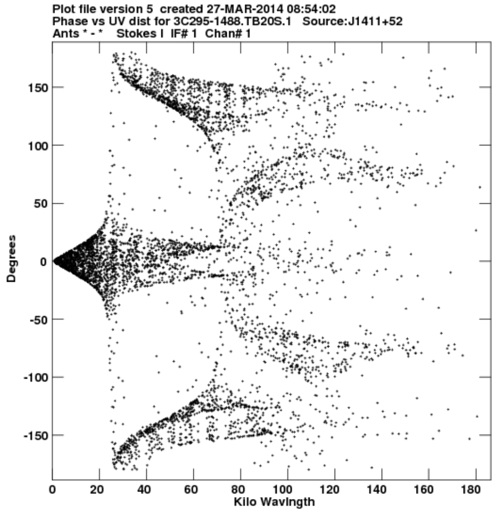




From the 1-d visibilities alone, we deduce

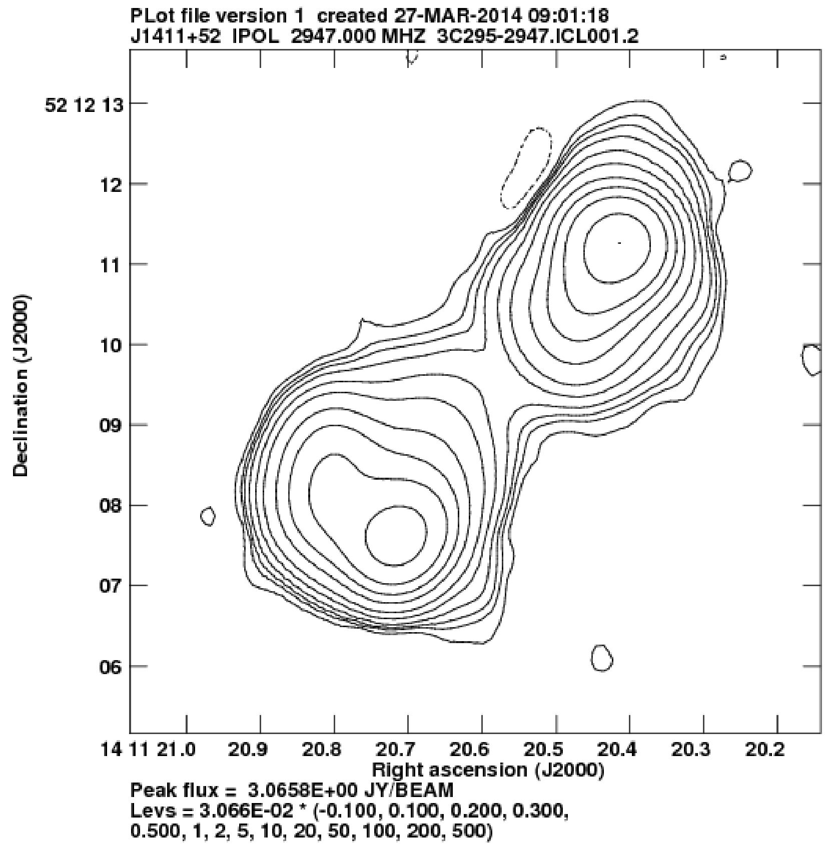
- The upper envelope scale of ~200 kλ says there is ~1” smaller scale structure present.
- The cyclical variations, with period of 50kλ says there is a pair of smaller objects, separated by about 4”
- The lack of an overall phase slope tells us the object is centered on the phase center.
- Without knowing the 2-d distribution of the phases and amplitudes, we can say nothing about the orientations.
- An unresolved component *may* be present, as the longest spacing visibility amplitudes remain high.





3C295 Image

- A double source, with separation of about four arcseconds.
- A width of 1 – 2 arcseconds.
- The resolution (psf) is not high enough to detect an unresolved component.
- (But, higher resolution data show the presence of compact 'hot spots' and a weak nucleus in the middle).



And That's It!

- As I noted in the beginning of the first lecture – these concepts are not complicated, but they are unfamiliar.
- The best (only?) way to become comfortable with the intricacies of interferometric imaging is to play with real data.
- Start with a simple source (calibrator), and use the visualization tools included in AIPS or CASA to 'look' at the data, and the resulting images.
- Above all – don't be afraid to ask questions of your more experienced colleagues.
- They won't know what you don't know if you don't ask!